The static general equilibrium model of the previous chapter features an exogenously specified capital stock, so that a savings decision of the household is excluded. In order to analyze the infratemporal choice of consumption, the life cycle model of savings is developed in this chapter. The following section introduces the most basic version without uncertainty and derives the optimal savings decision for old-age. Next, this basic model is extended to consider wage uncertainty and precautionary savings, interest uncertainty and optimal portfolio choice, lifespan uncertainty and annuity demand and finally myopic behavior with hyperbolic discounting. Note that throughout this chapter we follow a partial equilibrium approach, so that factor prices for capital and labor are exogenously specified and not endogenously determined as in the previous chapter.

5.1 Optimal savings in a certain world

In order to derive savings decisions it is assumed in the following that a household lives for three periods. In the first two periods he works and receives labor income \( w \) while in the last period he lives from his accumulated previous savings. In order to derive the optimal savings decisions \( s_1 \) and \( s_2 \) in both periods, the agent maximizes the utility function

\[
U(c_1, c_2, c_3) = u(c_1) + \beta u(c_2) + \beta^2 u(c_3)
\]

where \( \beta = \frac{1}{1+\delta} \) denotes a time discount factor (with \( \delta \) as the rate of time preference) and \( u(c) = \frac{c^{1-\gamma}}{1-\gamma} \) describes the preference function with \( \gamma \neq 1 \) as relative risk aversion. Households have no assets initially, so that their periodic budget constraints in the three periods are

\[
w = s_1 + c_1, \quad Rs_1 + w = s_2 + c_2 \quad \text{and} \quad Rs_2 = c_3,
\]

where \( R = 1 + r \) denotes the return on savings. After substituting the budget constraints into the utility function the maximization problem becomes

\[
\max_{s_1, s_2} U(s_1, s_2) = u(w - s_1) + \beta u(Rs_1 + w - s_2) + \beta^2 u(Rs_2).
\]

Program 5.1 shows how we can solve this problem using the subroutine \texttt{fminsearch} coming from the \texttt{minimization} module. We thereby store our model parameters in the
Program 5.1: Optimal savings in a certain world

```fortran
program household

...;

! lower and upper border and initial guess
low = (/0d0, 0d0/)
up = (/w, R*w+w/)
x = up/2d0

! minimization routine
call fminsearch(x, fret, low, up, utility)

! output
write(*,'(/a/)')' AGE CONS WAGE INC SAV'
write(*,'(i4,4f7.2/)')1,c(1),w,w,s(1)
write(*,'(i4,4f7.2/)')2,c(2),w,R*s(1),s(2)
write(*,'(i4,4f7.2)')3,c(3),0d0,R*s(2),0d0

end program

function utility(x)

! parameter module
use globals

implicit none

! variable declaration
real*8, intent(in) :: x(:)
real*8 :: utility

! savings
s = x

! consumption (insure consumption > 0)
c(1) = w - s(1)
c(2) = R*s(1) + w - s(2)
c(3) = R*s(2)
c = max(c, 1d-15)

! utility function
utility = -(c(1)**(1d0-gamma) + beta*c(2)**(1d0-gamma) +
        beta**2*c(3)**(1d0-gamma))/(1d0-gamma)

end function
```
module `global`. The solution to the problem is calculated using `fminsearch`. Recall that this subroutine takes a starting value $x$, a `real*8` value `fret` in which the function value in the minimum will be stored, a lower and upper bound for the optimization interval and the function that should be minimized as input variables. We define the lower bound as zero, i.e., households will not be allowed to run into debt. The upper bound is set such that savings can not exceed the maximum available resources in periods 1 and 2, respectively.

The function `utility` is the function to be minimized. We start by copying the input vector of the function to the savings decision $s$. Note that $s$ and $c$ are `real*8` arrays of length 2 and 3, respectively, that also are defined in the module `globals`. Thereby $s$ denotes the savings decision in the first two periods (savings in the last period have to be equal to zero) and $c$ consumption the the three periods of the life-cycle. Having copied the savings decisions, we can calculate consumption using the three budget constraints in (5.1). We limit consumption to positive values. If we didn’t do that, it might happen that consumption turns negative during the optimization process and we therefore run into an error. Finally, we calculate households utility multiplied by -1 and return the respective value. We finally can print our results on the console.

Not surprisingly, with $\beta = 1 / R$ we obtain the result

$$c_1 = c_2 = c_3 = \frac{2}{3}, \quad s_1 = \frac{1}{3} \quad \text{and} \quad s_2 = \frac{2}{3},$$

which could also be calculated taking first order conditions of the Lagrangian resulting from the utility function and the intertemporal budget constraint.

## 5.2 Uncertain labor income and precautionary savings

When income is certain, agents only build up savings in order to smooth consumption during the years of retirement when they have no labor income. This is the so-called *old-age savings motive*. If the interest rate equals the time preference rate, agents would not accumulate any savings in case they receive a certain income also in the last period of life. In reality, the income process during the life cycle is however much more disrupted. People are not able to work at the end of life (maybe due to health problems) and they receive a highly uncertain income in their middle years. This uncertainty of income in the second period gives rise to a second savings motive, so-called *precautionary savings*. Risk averse agents will then save more in the first period in order to dampen the volatility of their consumption in the second period.

We therefore assumed that wages in the second period $\tilde{w}$ are log-normally distributed with mean $\mu_w$ and variance $\sigma^2_w$, i.e., $\tilde{w} \sim \log N(\mu_w, \sigma^2_w)$. Figure 5.1 shows densities of log-normally distributed variables for three combinations of expectation and variance. The advantage of taking a log-normal than a normal distribution is that the log-normal distribution is bounded from below by zero. Hence, wages can not be negative. For our
simulations, we choose $\mu_w = w$ which equals the certain wage in the previous section. However, we will let the variance of the distribution differ.

The maximization problem of our household now changes to

$$\max_{s_1, \tilde{s}_2} E_1 U(s_1, \tilde{s}_2) = u(w - s_1) + \beta E_1 \left[ u(Rs_1 + \tilde{w} - \tilde{s}_2) + \beta u(R\tilde{s}_2) \right],$$

where $E_1$ denotes the expectation at the beginning of period 1 regarding earnings and consumption in the following periods. Obviously, there is only one amount of savings $s_1$ in the first period. Hence, with income being certain in this period, consumption and utility are deterministic and we have $c_1 = w - s_1$. In the second period, however, income is uncertain and therefore household may choose a different level of savings $\tilde{s}_2$ for any realization of income $\tilde{w}$. Hence, we are looking for a optimal function of savings $\tilde{s}(\tilde{w})$ in the second period. Given the density $\varphi$ of the distribution of $\tilde{w}$, we can write 

$$E_1 \left[ u(Rs_1 + \tilde{w} - \tilde{s}_2) + \beta u(R\tilde{s}_2) \right] = \int_0^\infty \varphi(\tilde{w}) \left[ u(Rs_1 + \tilde{w} - \tilde{s}_2(\tilde{w})) + \beta u(R\tilde{s}_2(\tilde{w})) \right] d\tilde{w}.$$

This gives rise to using a Gaussian quadrature method like implemented in the subroutine `normal_discrete` in the module `normalProb` in order to calculate expectations. However, we will come to that later.

The optimality condition for maximizing expected utility is that expected marginal utility be equated across periods, i.e.

$$u'(c_1) = E_1 u'(\tilde{c}_2) = E_1 u'(\tilde{c}_3) \quad (5.2)$$
where \( \beta = 1/R \) is assumed for simplicity. If marginal utility is convex (i.e. \( u'' < 0, u''' > 0 \)) we can show that expected marginal utility in period 2, i.e. \( E_1[u'(c_2)] \), exceeds marginal utility of expected consumption \( u'(E_1[c_2]) \), see Figure 5.2. This is because symmetric risk around an expected consumption value \( E_1(c_2) \) increases marginal utility more in the bad state than it reduces it in the good state. Consequently, expected marginal utility of period 2 increases with increasing risk. Consequently, compared to the case of

\[
\Delta s
\]

Figure 5.2: Expected marginal utility

certainty, marginal utility has to increase in period 1 when income in period 2 is uncertain in order to satisfy the first order condition (5.2). This means that consumption in period 1 has to decline which implies an increase in first-period savings \( s_1 \) compared to the certainty case in the previous section. The increase in \( s_1 \) due to uncertainty is the amount of precautionary savings the household makes. Figure 5.2 also clearly shows that precautionary savings increase when marginal utility becomes more convex (i.e. relative risk aversion \( \gamma \) increases) or when uncertainty increases due to a higher variance \( \sigma^2_w \) which increases the distance between \( c_1^2 \) and \( c_2^2 \).

With respect to savings \( s_2 \) in period 2 note that they are only stochastic at the beginning of period 1. When the actual savings decision in period 2 is made, uncertainty is gone since agents then already know the realization of their income. Consequently \( E_2u'(c_t) = u'(c_t), t = 2, 3 \).

After those theoretical considerations, we have to talk about how to solve this maximization problem. Program 5.2 shows how to do this. At first, we want to discretize the distribution of \( \tilde{w} \) in order to avoid calculating a whole integral. This can be done with the subroutine \texttt{log_normal_discrete} coming along with the module \texttt{normalProb}. Like the subroutine \texttt{normal_discrete}, \texttt{log_normal_discrete} generates \( n_w \) quadrature nodes \( w \) and weights \( \text{weight}_w \). The subroutine thereby makes use of the fact that a log \( N(\mu_w, \sigma^2_w) \)
Program 5.2: Optimal savings with wage risk

```fortran
program household2

  ...

  ! discretize log(wage)
  sig = log(1d0+sig_w/mu_w**2)
  mu  = log(mu_w)-0.5d0*sig
  call normal_discrete(w, weight_w, mu, sig)
  w = exp(w)

  ...

end program

function utility(x)

  ! parameter module
  use globals

  implicit none

  ! variable declaration
  real*8, intent(in) :: x(:)
  real*8 :: utility
  integer :: iw

  ! savings
  s(1, :) = x(1)
  s(2, :) = x(2:1+n_w)

  ! consumption (insure consumption > 0)
  c(1,:) = mu_w - s(1,1)
  c(2,:) = R*s(1,1) + w(:) - s(2,:)
  c(3,:) = R*s(2,:)
  c = max(c, 1d-15)

  ! expected utility of periods 2 and 3
  utility = 0d0
  do iw = 1, n_w
    utility = utility+weight_w(iw)*(c(2,iw)**(1d0-gamma)+ &
      beta*c(3,iw)**(1d0-gamma))
  enddo
  utility = -(c(1,1)**(1d0-gamma)+beta*utility)/(1d0-gamma)

end function
```
distributed random variable can be generated out of a normally distributed variable with mean and variance

\[ \mu = \log(\mu_w) - 0.5 \log \left( 1 + \frac{\sigma_{\bar{w}}^2}{\mu_w^2} \right) \quad \text{and} \quad \sigma^2 = \log \left( 1 + \frac{\sigma_{\bar{w}}^2}{\mu_w^2} \right). \]

Having discretized the distribution of \( \bar{w} \), we can calculate our expected utility function in periods 2 and 3 as

\[ E_1 \left[ u(Rs_1 + \bar{w} - \bar{s}_2) + \beta u(R\bar{s}_2) \right] \approx \sum_{i=1}^{n_w} \omega_{w,i} \left[ u(Rs_1 + w_i - s_{2,i}) + \beta u(Rs_{2,i}) \right]. \]

Note that now we also avoid the problem of calculating a whole optimal savings function \( \bar{s}_2(\bar{w}) \). Instead we only have to calculate one savings level \( s_{2,i} \) for any of the possible realizations of the discretized random variable \( w_i \).

Having discretized the distribution for \( \bar{w} \) we can again use \texttt{fminsearch} to minimize the utility of the household which is calculated in the function \texttt{utility}. This function again receives an input vector \( x \). However, the variables in \( x \) now have changed considerably compared to the previous section. The first entry still is savings from period 1 to period 2. The next \( n_w \) entries of \( x \) however denote savings in any possible realization \( w_i \) of the wage in period 2, i.e.

\[ x = (s_1, s_{2,1}, s_{2,2}, \ldots, s_{2,n_w})^T. \]

Consequently, \( x \) must be of length \( 1+n_w \). After having copied savings into a vector of dimensions \( 2 \cdot n_w \), we can calculate consumption in every period in every state of the world, i.e. any realization of wages. Note that savings in the first period are certain as household earns the certain average wage \( \mu_w \), i.e. we only need to use one savings level \( s(1,1) \). For convenience, however, we store the same savings and consumption level for every entry in dimension 2 of the arrays \( s \) and \( c \). Consumption in the second and third period is stochastic and depends on the realization of the wage and the savings levels in this state. Finally we can compute expected utility for periods 2 and 3 as shown above and add first periods utility afterwards. Having minimized the function utility, we can compute mean and standard deviations of consumption, income and savings in different periods. We will not show how to do this due to space restrictions. However, the respective functions and calculations can be seen from the program accompanying this book.

Table 5.1 shows the outcome of our simulations. We thereby again set risk aversion to 2, \( \beta = 1 \) and \( R = 1 \). We assume an expected wage of \( \mu_w = 1 \) and use \( n_w = 5 \) quadrature nodes and weights to approximate the log-normal distribution. We then vary the variance of the wage distribution holding its expectation constant. We find what we already concluded from our theoretical analysis. With increasing variance of the wage distribution
consumption in period 1 decreases and savings increase. This is due to the precautionary savings motive. In addition we find that savings and consumption in the second period always have same expectation and standard deviation. This is clear from the fact that, after the wage level is realized and there is no more uncertainty in the decision problem, household will react exactly in the same way as in the case of certain income, i.e. equally split consumption on periods 2 and 3. As \( c_3 = s_2 \) when the interest rate is zero, expectation and variance of \( c_2 \) and \( s_2 = c_3 \) have to be identical. In addition, the variance of consumption in period 2 always is one fourth of the variance of wages. Again this is clear if one recalls that \( \tilde{s}_2 = \tilde{c}_2 \). We therefore have

\[
\tilde{c}_2 = \frac{R s_1 + \tilde{w}}{2}
\]

from the second periods budget constraint and, as \( s_1 \) is deterministic, we obtain

\[
Var(\tilde{c}_2) = \frac{Var(\tilde{w})}{4}.
\]

### 5.3 Uncertain capital income

In the previous section we assumed capital income to be certain at a rate \( R = 1 \). In reality, however, we observe a lot of fluctuation in interest rates and asset return. It therefore seems natural to also let interest rates be stochastic in our model. Saving then turns out to be a risky investment.

Beneath wages which still are log-normally distributed in the second period we therefore also assume the return on capital \( \tilde{R} \) to be log-normally distributed with mean \( \mu_R \) and variance \( \sigma_R^2 \) and to be independent over time. Therefore \( R_2 \) and \( R_3 \), i.e. the return on capital in periods 2 and 3, respectively, are drawn from the same distribution but are independent.

<table>
<thead>
<tr>
<th>( \sigma_w^2 )</th>
<th>( c_1 )</th>
<th>( s_1 )</th>
<th>( E(c_2) )</th>
<th>( Std(c_2) )</th>
<th>( Var(c_2) )</th>
<th>( E(s_2) )</th>
<th>( Std(s_2) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.33</td>
<td>0.67</td>
<td>0.00</td>
<td>0.00</td>
<td>0.67</td>
<td>0.00</td>
</tr>
<tr>
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<td>0.39</td>
<td>0.69</td>
<td>0.22</td>
<td>0.05</td>
<td>0.69</td>
<td>0.22</td>
</tr>
<tr>
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<td>0.58</td>
<td>0.42</td>
<td>0.71</td>
<td>0.32</td>
<td>0.10</td>
<td>0.71</td>
<td>0.32</td>
</tr>
<tr>
<td>0.60</td>
<td>0.56</td>
<td>0.44</td>
<td>0.72</td>
<td>0.39</td>
<td>0.15</td>
<td>0.72</td>
<td>0.39</td>
</tr>
<tr>
<td>0.80</td>
<td>0.54</td>
<td>0.46</td>
<td>0.73</td>
<td>0.45</td>
<td>0.20</td>
<td>0.73</td>
<td>0.45</td>
</tr>
<tr>
<td>1.00</td>
<td>0.53</td>
<td>0.47</td>
<td>0.74</td>
<td>0.50</td>
<td>0.25</td>
<td>0.74</td>
<td>0.50</td>
</tr>
</tbody>
</table>

\( R = 1, \mu_w = 1, \gamma = 2, \beta = 1, n_w = 5. \)

Table 5.1: Precautionary savings motive when wages are risky
Our optimization problem now turns into

\[
\max_{s_1, \tilde{s}_2} E_1 U(s_1, \tilde{s}_2) = u(w - s_1) + \beta E_1 \left[ u(\tilde{R}_2 s_1 + \tilde{w} - \tilde{s}_2) + \beta u(\tilde{R}_3 \tilde{s}_2) \right].
\]

Obviously, \( \tilde{s}_2 \) now not only depends on the realization of the wage \( \tilde{w} \) but also on the return on capital of the second period \( \tilde{R}_2 \), so that we are looking for an optimal savings function \( \tilde{s}_2(\tilde{w}, \tilde{R}_2) \). In order to compute our model solution we again have to discretize our shock values regarding interest rates. Beneath weights and quadrature nodes \( \omega_{w,i} \) and \( w_i \), we therefore also compute weights and nodes \( \omega_{R,i} \) and \( R_i \) for the interest rate distribution and rewrite the expectation of utility in periods 2 and 3 as

\[
E_1 \left[ u(\tilde{R}_2 s_1 + \tilde{w} - \tilde{s}_2) + \beta u(\tilde{R}_3 \tilde{s}_2) \right] \approx \sum_{i=1}^{n_w} \sum_{j=1}^{n_R} \sum_{k=1}^{n_R} \omega_{w,i} \omega_{R,j} \omega_{R,k} \left[ u(R_js_1 + w_i - s_{2,i,j}) + \beta u(R_k s_{2,i,j}) \right].
\]

Program 5.3 shows how to solve the model with uncertain labor and capital income. The structure of the solution method is quite similar to the one in the previous section. In addition to weights and quadrature nodes for wages we also calculate weights \( \omega_{R,i} \) and nodes \( R_i \) for capital returns. Note that we now have one savings level \( s_1 \) in the first period and \( n_w \cdot n_R \) savings levels in the second period, as savings depend on the realization of income and capital returns. We again stack the different savings levels in the input vector \( x \) to the function \( \text{utility} \) in the following way:

\[
x = (s_1, s_{2,1,1}, s_{2,1,2}, \ldots, s_{2,1,n_R}, \ldots, s_{2,n_w,n_R})^T.
\]

After copying the values from \( x \) into our savings vector we can again calculate consumption for all possible realizations wages and interest rates. Note that consumption in the first period again is the same for any shock realization as it is chosen before households know about their income and capital returns. Having calculated consumption we again can compute expected utility in periods 2 and 3 according to (5.3). We then add first periods utility and multiply the result by \(-1\).

### 5.4 Uncertain capital income and portfolio choice

We now assume that the individual can split his investment between two different assets in both investment periods \( t = 1, 2 \). One is riskless (e.g. bonds) and yields a gross return of \( R \), the other one is risky (e.g. stocks) and has a gross return of \( \tilde{R}_t + 1 \) with mean \( \mu_R > R \) being the average return on equity. The joint distribution of labor income and capital return in period 2 is a two dimensional log-normal distribution

\[
\begin{bmatrix}
\tilde{w} \\
\tilde{R}_2
\end{bmatrix} \sim \log N \left( \begin{bmatrix}
\mu_w \\
\mu_R
\end{bmatrix}, \begin{bmatrix}
\sigma_w^2 & \rho \sigma_w \sigma_R \\
\rho \sigma_w \sigma_R & \sigma_R^2
\end{bmatrix} \right),
\]

where \( \sigma_w \) and \( \sigma_R \) are the standard deviations of the labor income and capital return, respectively, and \( \rho \) is the correlation coefficient.
Program 5.3: Optimal savings with wage risk and risky capital returns

```plaintext
function utility(x)

    ...

    ! savings
    s(l,:,,:) = x(l)
    ic = 2
    do iw = 1, n_w
        do ir2 = 1, n_R
            s(2, iw, ir2) = x(ic)
            ic = ic+1
        enddo
    enddo

    ! consumption
    c(l,:,,:) = mu_w - s(1,1,1)
    do iw = 1, n_w
        do ir2 = 1, n_R
            c(2,iw,ir2,:) = R(ir2)*s(1,1,1) + w(iw) - s(2,iw,ir2)
            do ir3 = 1, n_R
                c(3,iw,ir2,ir3) = R(ir3)*s(2,iw,ir2)
            enddo
        enddo
    enddo
    c = max(c, 1d-15)

    ! expected utility of periods 2 and 3
    utility = 0d0
    do iw = 1, n_w
        do ir2 = 1, n_R
            do ir3 = 1, n_R
                prob = weight_w(iw)*weight_R(ir2)*weight_R(ir3)
                utility = utility + prob*(c(2,iw,ir2,1)**(1d0-gamma) + &
                                          beta*c(3,iw,ir2,ir3)**(1d0-gamma))
            enddo
        enddo
    enddo
    utility = -(c(1,1,1,1)**(1d0-gamma)+beta*utility)/(1d0-gamma)
end function
```

where \( \rho \) represents the correlation between the two. For simplicity, we assume the distribution of \( \tilde{R}_3 \) to have the same mean and variance as that of \( \tilde{R}_2 \), but to be independent of both \( \tilde{w} \) and \( \tilde{R}_2 \). Let \( \alpha_t \) be the share of agent’s portfolio being invested in risky assets. Consequently, the return on the portfolio is given by

\[
\tilde{R}_{t+1}^\nu = \alpha_t \tilde{R}_{t+1} + (1 - \alpha_t)R = R + \alpha_t (\tilde{R}_{t+1} - R).
\]
The mean portfolio return is 
\[ E_t \left[ R_{t+1}^p \right] = R + \alpha_t (\mu_R - R) \]
where \( \mu_R - R \) defines the risk premium. The variance is 
\[ \text{Var} \left[ R_{t+1}^p \right] = \alpha_t^2 \sigma_R^2. \]

Besides deciding about the optimal savings amount, the individual now also has to optimally allocate a proportion \( \alpha_t \) of his savings to risky assets in periods 1 and 2. Consequently, the periodic budget constraints change to

\[
\begin{align*}
  c_1 + s_1 &= w \\
  \tilde{c}_2 + \tilde{s}_2 &= \tilde{w} + [R + \alpha_1(\tilde{R}_2 - R)]s_1 \\
  \tilde{c}_3 &= [R + \alpha_2(\tilde{R}_3 - R)]\tilde{s}_2.
\end{align*}
\]

Program 5.4 shows part of the function that is needed to compute the solution of the portfolio choice problem. Before minimizing this function we have to discretize the two-dimensional log-normal distribution. This can again be done with the routine \text{log\_normal\_discrete}.

In the case of a two-dimensional distribution, however, this subroutine receives an array of integer values defining the number of points to use in each direction as first argument. We then pass an array of dimension \( n_w \cdot n_R \times 2 \) and one of dimension \( n_w \cdot n_R \). In the former the subroutine will store the \( \tilde{w} \) and \( \tilde{R}_2 \) quadrature nodes, the latter represents the respective weights. The moments of the approximated distribution can then be calculated as shown in the full program code. Compared to the approximation of independent variables in the previous section, we now do not get \( n_w \) quadrature nodes for \( \tilde{w} \) and \( n_R \) nodes for \( \tilde{R}_2 \). This is due to the possible correlation of the two variables. If the variables were e.g. positively correlated, then at least in tendency we can see that the larger \( \tilde{w} \), the larger \( \tilde{R}_2 \). This fact can only be matched by assigning any approximation value \( w_i \) of \( \tilde{w} \) a separate set \( R_{ij}, j = 1, \ldots, n_R \) of approximation values for \( \tilde{R}_2 \). In the array \( wR \) we therefore will have the entries

\[
wR = \left[ \begin{array}{cccccccc}
  w_1 & w_1 & \ldots & w_1 & w_2 & \ldots & w_{nw} & \ldots & w_{nw} \\
  R_{11} & R_{12} & \ldots & R_{1n_R} & R_{21} & \ldots & R_{n_w,1} & \ldots & R_{n_w,n_R}
\end{array} \right]^T.
\]

As the number of choice variables now has increased dramatically, we adjust the tolerance level of our minimization routine to \( 10^{-14} \). This can be done using the subroutine \text{settol\_min} in the module \text{minimization}. Before starting \text{fminsearch}, we still have to set lower and upper bound for minimization. Note that we now have an overall number of choice variables of \( 2 \cdot (1 + n_w \cdot n_R) \), i.e. a savings amount \( s \) and a portfolio composition \( \alpha \) for period 1 and any wage and interest rate combination in \( wR \) of period 2. As we don’t want to allow for short selling of neither bonds nor equity, we have \( \alpha_t \in [0, 1] \).

We can now start minimizing the function \text{utility} part of which is shown in Program 5.4. After having declared variables and modules, we start with copying savings levels and portfolio compositions from the input array \( x \). Note that we again have to stack all the decision variables into one array for the optimizer to work properly. With the savings levels and portfolio compositions we can calculate consumption in the different periods and approximation values for \( \tilde{w} \), \( \tilde{R}_2 \) and \( \tilde{R}_3 \) using the constraints in (5.5). Obviously,
there is only one consumption level in period 1 and \( n_w \cdot n_R \) levels in period two, as in the second period, the realization of \( \tilde{R}_3 \) is not yet revealed. In the third period, we finally have
5.4. Uncertain capital income and portfolio choice

\(n_w \cdot n_R \cdot n_R\) different levels depending on the realizations of the three random variables. Last but not least we have to compute individuals utility. The probability for any of the \(n_w \cdot n_R \cdot n_R\) states to occur is the product of the probability for a certain \(\bar{w}\) and \(\bar{R}_2\) combination and that for a realization of \(\bar{R}_3\) due to the independency of \(\bar{R}_3\) from the other random variables.

In order to analyze individual portfolio choice, we assume in the first simulation in Table 5.2 that wage income and interest rates are certain, i.e. \(\sigma^2_R = \sigma^2_w = 0\). We set the expected return on equity to 1.22, which results in a risk premium of 0.22. If we assume one period to cover about 20 years, this figure amounts to an annual premium of about 1%. This seems fairly low but reasonable with the low risk aversion parameter of 2. While the re-

<table>
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<th>(\sigma^2_w)</th>
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</tr>
</tbody>
</table>

\(R = 1, \mu_R = 1.22, \mu_w = 1, \gamma = 2, \beta = 1, n_R = 5, n_w = 5\).

Table 5.2: Uncertain income and individual portfolio choice

turn on bonds remains at \(R = 1.0\), stocks now yield a certain return of \(R_2 = R_3 = 1.22\), so that people will only save in stocks (i.e. \(\alpha_1 = \alpha_2 = 1.0\)). Since the discount rate is kept at zero and the increase in the interest rate is fairly modest, savings in both periods remain the same as in the certainty case of the previous section, but the consumption level increases with rising age in all simulations. The second simulation introduces risky stock investment but keeps wages certain. As a consequence, the portfolio composition changes noticeably. While in the first period, agent still invests everything in equity, in the second period, the equity share decreases to about 0.33. This is due to the fact that our household is of constant relative risk aversion. With this preference structure, he will always put the same fraction of his invested wealth into equity. The fraction then only depends on the risk structure and equity premium of the portfolio, but not on the invested amount. In the first period, invested wealth however not only consists of savings but also of human capital, i.e. the wage that is payed in period 2. As this wage is certain, household already has a large riskless "investment". Consequently, he puts the remaining invested amount, namely all his savings, into equity. He would even prefer to sell short bonds in order to finance further equity investment. This however, is not possible due to the imposed short-selling constraints.\(^{14}\) In the second period, invested

\(^{14}\) Try to show this behavior by loosening the upper constraint for \(\alpha\). You will see that \(\alpha_1 > 1\) will then be optimal.
wealth only consists of savings and therefore the share invested in equity decreases to 0.33. With the constant relative risk aversion specification, this share is the same for all possible realizations of $\tilde{R}_2$. Consequently, the standard deviation of $\tilde{\alpha}_2$ is 0. Due to the decreased equity share in the portfolio in period 2, the expected rate of return decreases from 1.22 to $0.67 \cdot 1 + 0.33 \cdot 1.22 = 1.07$. In order to compensate for the resulting decrease in expected consumption in the third period, savings in periods 1 and 2 increase. Next, we introduce uncorrelated wage uncertainty, so that the uncertainty of second period income increases. Now, human capital, i.e. the wage in the second period, already is a risky investment. Hence, the share of equity in the first periods portfolio has to decrease to $\tilde{\alpha}_1 = 0.75$. Due to the precautionary savings motive that was already introduced in Section 5.2, overall savings in the first period will rise considerably. Since the risk structure of second period’s investment is not affected by wage uncertainty and preferences are of constant relative risk aversion type, $\tilde{\alpha}_2$ remains unaltered. Note, that the equity share in the portfolio still decreases over the life cycle. This pattern however is turned around in the next simulation, where we assume that wage and interest income risk are positively correlated. Consequently, a bad realization of wage income is very much likely accompanied by a low interest realization. In order to avoid situations with low wage and capital income, all savings in the first period are now invested in bonds. This again reduces the expected portfolio return and therefore overall savings increase in the first period. Since the risk structure of capital returns in the third period again is not influenced by the correlation, $\tilde{\alpha}_2$ remains the same. Note that now the equity share in the portfolio increases over the life cycle. Of course, the opposite happens when wage and interest risk are negatively correlated. In this case, it is possible to hedge against wage income risk by investing in stocks, as low wage income is most likely to come along with high interest rates. Consequently, precautionary savings decrease and all first-period savings are invested in stocks.

5.5 Uncertain life span and annuity choice

Another important source of risk households face over the life cycle is the risk of early death. Annuities are savings contracts that may insure agents against this risk. Such contracts pay out a lot more compared to regular assets in the case household survives, however, if he dies, the insurer receives the remaining amount of savings. At first sight, one would expect annuity contracts to be the preferred savings vehicle for households facing life span uncertainty. Nevertheless, taking a look at the data, we find that people save much more in regular assets than in annuities. In the literature, this fact is called the annuity puzzle. Various attempts were made to explain this behavior. Bequest motives or market failure due to adverse selection problems might be one explanation. However, as we will show in this section, labor income uncertainty can also explain part of the non-annuitization of assets, especially in early stages of life.

In the following, we will therefore construct a model of uncertain life span. Household
only survive with probability $\pi_2$ from period 1 to 2 and, conditional on having survived to period 2, he only lives up to period 3 with probability $\pi_3$. In order to insure against the risk of early death, he might purchase annuities. An annuity contract bought in period $t = 1, 2$ will have a constant payout stream in all future periods. The expected present value of payments as of time $t$ consequently is

$$h_1 = \frac{\pi_2}{R} + \frac{\pi_2 \pi_3}{R^2} \quad \text{and} \quad h_2 = \frac{\pi_3}{R}.$$

If we assume a perfectly competitive annuity market, insurers will not run any surplus. Therefore, for one unit of income invested in annuities at time $t$, household will receive a constant income stream of $\frac{h_t}{R}$ in all future periods.

Beneath choosing the optimal asset level, agent now also has to decide about which fraction $\alpha_t$ of his savings to invest in annuities. The remaining part of assets $1 - \alpha_t$ will be invested in regular assets at a fixed rate $R$. Hence, the optimization problem turns into

$$\max_{s_1, \alpha_1, \tilde{s}_2, \tilde{\alpha}_2} u(c_1) + \pi_2 \beta E[u(\tilde{c}_2) + \pi_3 \beta u(\tilde{c}_3)],$$

where expectations now not only are formed with respect to labor income uncertainty but also with respect to survival. Household therefore only receives utility in a certain period if he survives. We assume that savings have to be greater than a lower threshold $s$. With this assumption, we can loosen households borrowing limit of $s \geq 0$ imposed in the previous sections. The budget constraints finally turn into

$$c_1 = w - s_1$$

$$\tilde{c}_2 = \tilde{w} + (1 - \alpha_1)s_1 + \frac{\alpha_1 s_1}{h_1} - \tilde{s}_2$$

$$\tilde{c}_3 = p + (1 - \tilde{\alpha}_2)s_2 + \frac{\tilde{\alpha}_2 \tilde{s}_2}{h_2} + \frac{\alpha_1 s_1}{h_1},$$

where $p$ is a pension that is payed out to the individual in the last period of life.

Program 5.5 shows part of the function that is needed to compute the solution of the annuitization problem. Before minimizing this function, we again discretize the wage distribution into interpolation nodes and weights and set the tolerance level to $10^{-14}$. Next we compute the annuity factors $h_t$ and set the intervals for optimization. Note that in this program we set the lower limit of optimization for $s_t$ to $s$. However, we still assume that $0 \leq \alpha_t \leq 1$ has to hold. This means that agents will be allowed to run into debt if $s < 0$, however, they will not be able to short-sell regular assets in order to finance annuity purchases. This is a necessary condition for the existence of a solution, as in the second period of life, annuities are strictly preferred to regular assets. Consequently, households would always like to borrow from regular assets and invest in annuities as this would be an arbitrage strategy. Having specified the optimization intervals, we can start minimizing the function utility. In this function, we again first copy savings levels and annuity shares in the portfolio into the respective variables. After that we calculate consumption in the different periods and expected utility. Note that beneath the weights for the wage distribution, we also have to compute expectations using survival probabilities.
Program 5.5: Lifespan uncertainty and annuity choice

```plaintext
function utility(x)

    ...

    ! savings
    s(1,:) = x(1)
    alpha(1,:) = x(2)
    ic = 3
    do iw = 1, nw
        s(2,iw) = x(ic)
        alpha(2,iw) = x(ic+1)
        ic = ic+2
    enddo

    ! consumption (insure consumption > 0)
    c(1,:) = mu_w - s(1,1)
    c(2,:) = R*(1d0-alpha(1,1))*s(1,1) + alpha(1,1)*s(1,1)/h(1) + & w(:,)-s(2,:)
    c(3,:) = R*(1d0-alpha(2,:))*s(2,:) + alpha(2,:)*s(2,:)/h(2)+ & alpha(1,1)*s(1,1)/h(1)+pen
    c = max(c, 1d-20)

    ! expected utility of periods 2 and 3
    utility = 0d0
    do iw = 1, nw
        utility = utility + weight_w(iw)*(c(2,iw)**(1d0-gamma)+ & pi(3)*beta*c(3,iw)**(1d0-gamma))
    enddo

    ! add first period utility
    utility = -(c(1,1)**(1d0-gamma)+pi(2)*beta*utility)/(1d0-gamma)

end function
```

Table 5.3 shows some simulation results of the model. We set survival probabilities at \( \pi_2 = 0.8 \) and \( \pi_3 = 0.5 \) and assume a pension of \( p = 1 \). If in the first simulation wages are certain at a value of \( \mu_w = 1 \) in both periods, households will not save neither in regular assets nor in annuities, as their pension is enough to finance old age consumption. If we now increase wage uncertainty, we see that overall savings steadily increase. This is due to the precautionary savings motive. If uncertainty is low, agents start to invest in annuity contracts, as those are the preferred savings vehicle due to higher returns. However, with increasing wage risk, it turns out that households start to put their savings in regular assets rather than annuities in the first period. This behavior is quite reasonable: annuities bought in the first period only pay out part of the savings amount in period 2. As the need for precautionary savings to insure a minimum consumption level for bad realizations of wage income increases, households would like to transfer more income into the second
5.6. Time inconsistent preferences: the hyperbolic discounter

In recent experimental studies and field experiments it is often shown that the model of the rational exponential utility maximizing agent does not fit individual behavior very good. Often, people rather behave in a time inconsistent way. When it comes to savings decisions, for example, people tend to consume too much in presence and consequently do not have as much resources as they would want to have in the future.

A way of modeling time inconsistent behavior in economic models is that of hyperbolic discounter.

### Table 5.3: Uncertain life span and annuity choice

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<td>0.12 1.02</td>
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</tbody>
</table>

$R = 1, \mu_w = 1, \gamma = 2, \beta = 1, \pi_2 = 0.8, \pi_3 = 0.5, n_w = 5.$

Period without increasing third period consumption too much. This can only be done by saving in regular assets. In the extreme case of a very high income uncertainty in period 2, agents will only save in regular assets in period 1. Note that for period 2 the dominant savings strategy is $\alpha_2 = 1$, as the return on annuities is higher than that of regular assets and the full savings amount will be payed out in the next period. In the last simulation of Table 5.3, we loosen household’s borrowing constraint by setting $\xi$ to $-\infty$. Without borrowing constraints, the full savings amount in period 1 will again be invested in annuities. As agents can now borrow against future annuity income, they are able to transfer income from period 3 into period 2. Therefore they don’t mind that part of their annuity savings will be payed out in period 3 and annuities again become the dominant savings vehicle. Note, however, that the assumption of no borrowing constraints at later stages in life is quite unrealistic, since an annuity can’t be seen as a real collateral. If agent’s die early, the insurer that issued the annuity will receive all the capital left in the contract. Hence, the bank will not be repayed. Therefore we think that borrowing constraints might be a reasonable assumption especially in later stages of life.

15 In the program we set $s_{\text{lower}}$ to $-1$ in order to avoid computational problems. However, we have checked that no agent is borrowing constraint in this simulation.
discounting agents. In a three period model, a time hyperbolic consumer maximizes the utility function
\[ u(s_1, \hat{s}_2) = u(w_1 - s_1) + \delta \beta [u(Rs_1 + w_2 - \hat{s}_2) + \beta u(R\hat{s}_2)], \]
where we already plugged in the periodic budget constraints. \(\delta\) hereby is the hyperbolic discount factor. Note that this factor, unlike the regular time discount factor, applies to any future period in the same way, i.e. any future period is given the additional weight \(\delta\). Up to now, there is no time inconsistency in this behavior. However, when it comes to the savings decision in the second period of life, agent will maximize the utility function
\[ u(s_2) = u(Rs_1 + w_2 - s_2) + \delta \beta u(Rs_2), \]
i.e. he will again apply the discount factor \(\delta\) to the future period although this additional discounting was not present in the first period’s optimization problem. If it was, \(\delta\) would simply be an additional discount factor and agent should maximize
\[ u(w_1 - s_1) + \delta \beta [u(Rs_1 + w_2 - s_2) + \delta \beta u(Rs_2)] \]
in period 1. The question now is what household believes in period 1 about how much he will save in period 2. This determines his prediction of future savings \(\hat{s}_2\) and influences his savings decision in the first period. In general, household will form his believes according to the maximization problem
\[ \max_{\hat{s}_2} u(Rs_1 + w_2 - \hat{s}_2) + \delta \beta u(R\hat{s}_2). \]
In the literature one usually distinguished two cases: the naive consumer thinks that he will behave in a perfectly rational way in future periods and therefore forms his believes using \(\hat{\delta} = 1\). The naive consumer will therefore never realize that his behavior is time-inconsistent and will always make the same mistakes in savings decisions. The sophisticated consumer, on the other hand, perceives his time-inconsistent behavior and therefore forms believes with \(\hat{\delta} = \delta\). Obviously, there is not much a sophisticated consumer can do against his time-inconsistent behavior. However, sometimes hyperbolic agents have access to some commitment device. A commitment device is a contract that allows agents to commit themselves in period 1 to certain future consumption levels. Popular commitment devices e.g. are annuities or retirement plans. Those plans usually are withdrawal restricted during the contribution phase and pay out certain fixed income streams in the payment phase. Hence, households can intentionally transfer part of their savings to far away future periods and therefore commit themselves to save more.

In our problem, we introduce a simple withdrawal restricted savings contract that can be bought in period 1 up to an upper limit of \(\bar{s}_1\). In opposite to regular savings, this contract will pay out in period 3 first. We assume that this contract causes some costs and therefore the return will be \(\kappa R^2\). With this contract, the decision problem of a hyperbolic discounter in period 1 can be written as
\[ \max_{s_1, s_1'} u(w_1 - s_1 - s_1') + \delta \beta [u(Rs_1 + w_2 - \hat{s}_2) + \beta u(R\hat{s}_2 + \kappa R^2 s_1')] \] (5.6)
subject to the constraint that $\hat{s}_2$ is the result of the maximization problem

$$\max_{\hat{s}_2} u \left( R s_1 + w_2 - \hat{s}_2 \right) + \hat{\delta} \beta u \left( R \hat{s}_2 + \kappa R^2 s'_1 \right).$$

Program 5.6 shows the functions that are needed to compute the solution to the time-inconsistent decision problem. The function \texttt{utility} that describes utility in the first period receives the amount of regular savings $s_1$ and restricted savings $s'_1$ as an input. From this savings levels and the wage in period 1, we can compute consumption in the first period. In order to get consumption levels in the next two periods, we have to solve the separate second period optimization problem described in (5.7). We therefore use the module \texttt{minimization} which is an exact copy of \texttt{minimization} but only has a different name. We need this new module as we want to use the subroutine \texttt{fminsearch} in a nested way and it is not possible to call the same subroutine twice for the first period and the second period optimization problem, respectively. Having minimized the function \texttt{utility} that allows us to form believes about future savings and consumption levels either using $\hat{\delta} = \delta$ or $\hat{\delta} = 1$, we can calculate agent’s utility in the first period and calculate optimal savings in regular and restricted accounts using \texttt{fminsearch} from the module \texttt{minimization}.

The problem of determining the optimal levels of $s_1$ and $s'_1$ for the sophisticated consumer is a problem with several local optima. This can be seen from Figure 5.3. In this figure we show the utility of a sophisticated hyperbolic household in period 1 with a fixed savings level of $s_1 + s'_1 = 0.45$. On the abscissa we plot $s'_1$, so that $s_1 = 0.45 - s'_1$. The red line depicts optimal savings in the second period. Note that at $s'_1 \approx 0.4$, regular savings in period 2 will be zero as all restricted savings are intended for the last period only. Actually,
Program 5.6: Hyperbolic discounting

\begin{verbatim}
function utility(x)
    ... ! savings
    s(1) = x(1)
    sr = x(2)

    ! consumption (insure consumption > 0)
    c(1) = w(1) - s(1) - sr
    c(1) = max(c(1), 1d-20)

    s_help = (R*s(1)+w(2))/2d0
    call fminsearch(s_help, fret, 0d0, R*s(1)+w(2), utility2)

    ! utility function
    utility = -(c(1)**(1d0-gamma) + delta*beta*c(2)**(1d0-gamma) +&
                delta*beta**2*c(3)**(1d0-gamma))/(1d0-gamma)
end function

function utility2(x)
    ... ! savings
    s(2) = x

    ! consumption (insure consumption > 0)
    c(2) = R*s(1) + w(2) - s(2)
    c(3) = R*s(2) + R**2*sr*kappa
    c(2:3) = max(c(2:3), 1d-20)

    ! utility function
    if(soph) then
        utility2 = -(c(2)**(1d0-gamma) +&
                    delta*beta*c(3)**(1d0-gamma))/(1d0-gamma)
    else
        utility2 = -(c(2)**(1d0-gamma) +&
                    beta*c(3)**(1d0-gamma))/(1d0-gamma)
    endif
end function
\end{verbatim}

people would like to borrow against their future restricted savings income. However, the borrowing constraint imposed in the model prevents this behavior. The green line shows $u(s_1, s'_1)$ for the different savings combinations. At first this utility decreases. This is clear
if we take a look at the savings decision in period 2 again. If household invests a small amount in the restricted account, in period 2 he will reduce regular savings by exactly $R$ times the amount saved. Hence, there is no influence on total assets held at the beginning of period 3. As restricted accounts are costly, i.e. $\kappa < 1$, household looses by increasing $s'_{1}$, as the return on savings shrinks. Nevertheless, at the moment $s_{2}$ is equal to zero and the agent is borrowing constraint in period 2, utility noticeably rises with increasing $s'_{1}$. This is due to the fact that now restricted savings work as a commitment device. Since household is not able to borrow in period 2, with restricted savings he can transfer some additional resources into period 3 without having himself reduce savings in 2 by the same amount. Consequently, resources in period 3 increase and utility rises as the problem of under-saving is reduced.

In order to solve this problem with multiple equilibria, we use the following algorithm: We partition the interval $[0, 0.99w_{1}]$ into $n$ different intervals

$$I_{k} = \left[0.99w_{1}\frac{k-1}{n}, 0.99w_{1}\frac{k}{n}\right], \quad k = 1, \ldots, n.$$ 

For each interval $I_{k}$ we search for the optimal solution $(s_{1,k}, s'_{1,k})$ of the problem in (5.6) and (5.7) with $s_{1} \geq 0$ and $s'_{1} \in I_{k}$ using fminsearch. We choose that combination that guarantees the highest out of the $k$ resulting utility levels. Note that this algorithm is quite similar to the one presented in Section 3.3.3. Last but not least, for a naive consumer after having computed first period decisions and his believes about second period savings, we have to calculate his actual behavior in period 2 using the function utility with a hyperbolic discount factor of $\delta$. We can finally output agents planned and actual behavior for the different periods.

Some simulation results from this model are shown in Table 5.4. We use similar param-

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$R = 1, \kappa = 0.99, w_{1} = 1, w_{2} = 0.5, \gamma = 2, \beta = 1.$

Table 5.4: Hyperbolic discounting and commitment devices
In the first row, we see the behavior of a rational consumer in this model. As the discount factor is 1 and the interest rate 0, household wants to consume the same amount in every period. Therefore, he transfers 0.5 of his income in period 1 into period 3. In the next simulation, we see the results for a naive hyperbolic consumer. This consumer saves less in the first period due to the additional discount factor. In period 1 he plans to split his available resources in period 2, i.e. \( R s_1 + w_2 = 0.88 \), evenly across the two remaining periods. However, when in period two, we find that he actually will save much less than the intended amount of 0.44. This illustrates the time-inconsistent behavior of such agents. Up to now, we assumed that agents were not allowed to save in restricted accounts, i.e. \( \sigma^r_1 = 0 \). In the program code, we can impose this restriction by setting the parameter \( \text{restr} \) to 0. If we set this parameter to 1, agents will be allowed to save as much as they want in restricted accounts. Nevertheless, naive consumers will not use this savings vehicle, as they believe to behave fully rational in period 2. In their opinion, restricted savings only decrease utility as they have a slightly lower return. Consequently, their behavior doesn’t change compared to the previous simulation. In the next row, we let people be sophisticated. Recall that sophisticated agents foresee that they will under-save in period 2. When these agents are not allowed to save in restricted accounts, they behave in a very similar fashion as the naive consumer, except for the fact that their believes and actual savings levels are identical. However, if we give these households access to restricted accounts, they will actually make use of them and commit themselves against under-saving in period 2. Note that the savings level in restricted accounts is not exactly equal to the intended savings level of the naive consumer. This is due to the slightly lower return of restricted savings compared to regular ones. In terms of utility levels, we find that the commitment effect that comes with restricted accounts increases utility in all three periods of the sophisticated consumer’s life.