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Optimizing Multi-Stage Production with Constant Lot Size and Varying Number of Unequal Sized Batches

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Proof of Convexity of Total Cost Function

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In the following stage specific subscripted symbols refer to stage s of a production system with s = 1, 2,..., S stages.

c_s	=	unit inventory holding cost per unit of time at stage s
$C(Q, M)$	=	total cost function subject to the lot size Q and the vector of the stage specific transportation frequencies M
$C_s(Q, m_s)$	=	total cost function with respect to two adjacent stages s and s + 1
d	=	constant demand rate $d = P_{S+1}$
D	=	total demand in the planning period $(D = d \cdot T)$
m_s	=	total number of batches at stage s
M	=	$\{m_1; m_2; ...; m_S\}$, a vector
$\max(P)_{s,s+1}$	=	maximum of P_s and P_{s+1}
$\min(P)_{s,s+1}$	=	minimum of P_s and P_{s+1}
P_s	=	constant production rate at stage s
q_i^s	=	size of batch number i at stage s
q_{Min}^s	=	size of the smallest batch at stage s
Q	=	lot size
S_s	=	setup cost per lot at stage s
T	=	length of the planning period
T_s	=	transportation cost of one batch from stage s to stage s+1

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1. Introduction

Lot sizing plays an important role in manufacturing planning, especially in series production. In this paper the term *lot* denotes the quantity of a product manufactured without interruption by other products on a specific facility. Each changeover from one type of product to another causes an interruption of the manufacturing process, because the shared facilities have to be set up anew. Considered in this paper is a multi-stage production environment.

The units of a product fabricated at a specific facility (stage) have to be transported to the succeeding manufacturing stage or to the sales area, respectively. The term *batch* represents that portion of a production lot which is conveyed simultaneously to the following stage. However, with respect to determinstic lot size models the literature in general ignored the variety of alternatives of how to transport a lot between adjacent stages. Most often, it is assumed that either complete lots are shipped or that each (infinitesimal) item of a lot is conveyed immediatly after its completion.

Models that consider the simultaneous optimization of the production lot size and the corresponding transportation batch sizes can be distinguished into two classes:

- Only equal sized batches can be transported between succeeding stages.
- Unequal sized batch shipments are allowed.[1]

We consider a determinstic lot size model for a single product produced in a serial manufacturing system with an unrestricted number of stages. The main characteristics are described as follows:

- *All parameters are constant and deterministic within the planning period.*
- *Capacity constraints of the production system are considered as not relevant.*
- *No backlogging (deliberate shortage) is permitted.*
- *A uniform lot size is manufactured through all stages.*

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1 For lot sizing models related to transportation activities, readers are referred to Bogaschewsky/Buscher/Lindner, Optimizing Multi-Stage Production, 2001; Bogaschewsky/Buscher/Lindner, Simultanplanung, 1999 and the references therein.

- *Transportation of batches to the following stage is allowed before the whole lot is completed at the respective stage. Each batch transport causes the same fixed (transportation) cost for any amount of items in the batch.*
- *Each lot is produced at each stage with only one setup and without interruption.*
- *The batch sizes between adjacent stages follow a geometric series.*
- *Linear inventory holding costs are assumed at all stages. The cost of holding one unit of process inventory may differ from stage to stage.*
- *The units of the considered product are infinitely divisible.*
- *Setup and transportation times are insignificant and hence ignored.*
- *The rate of continuous demand at sales is lower than the slowest manufacturing rate for a product type through all stages.*

For a production system with an unrestricted number of stages the total cost function is given by:

$$C(Q,M) = \sum_{s=1}^{S} \frac{Q \cdot \gamma_S \cdot D}{(\delta_s)^{m_s} - 1} + \frac{Q}{2} \cdot \left(\frac{1}{\min(P)_{s,s+1}} - \frac{1}{\max(P)_{s,s+1}} \right) \cdot c_S \cdot D + \\ + \sum_{s=1}^{S} (S_s + T_s \cdot m_s) \cdot \frac{D}{Q} \qquad (1)$$

where:

$$\gamma_S = \frac{c_S \cdot (\delta_S - 1)}{\max(P)_{s,s+1}} \quad \text{for } s = 1, 2, \ldots, S \quad \text{and} \quad \delta_s = \frac{\max(P)_{s,s+1}}{\min(P)_{s,s+1}} \quad \text{for } s = 1, 2, \ldots, S.$$

Requirements:

All variables and parameters must be greater than zero, $P_S \neq P_{S+1}$ for s = 1, 2,..., S, $P_S > d$ for s = 1, 2,..., S and $m_S \geq 1$ for s = 1, 2,..., S.

The first term of the cost function represents the inventory holding costs over all stages. The second term adds up the fixed costs (both, set-up and transportation of the m_S batches) per stage. Optimization procedures for two different planning situations (with constant or variable m_S-values, respectively) that minimize cost function (1) are described in:

- Bogaschewsky, R. /Buscher, U. /Lindner, G.: Optimizing Multi-Stage Production with Constant Lot Size and Varying Number of Unequal Sized Batches, in: *Omega, 2001* (to appear).

- Bogaschewsky, R. /Buscher, U. /Lindner, G.: Simultanplanung von Fertigungslosgröße und Transportlosgrößen in mehrstufigen Fertigungssystemen – Zwei statisch deterministische Ansätze bei unrestringierten Kapazitäten, Arbeitsbericht des Lehrstuhles für Betriebswirtschaftslehre, insbesondere Produktionswirtschaft, Dresdner Beiträge zur Betriebswirtschaftslehre, Nr. 30, 1999.

The remainder of this paper is organized as follows. In section 2 we prove that the cost function C(Q, M) is convex. Convexity of C(Q,M) ensures that the solutions obtained when applying one of the two procedures mentioned above are optimal. The proof in section 2 is based on a specific inequality assumption. Section 3 shows that this assumption holds true for our case.

2 Proof of Convexity of Total Cost Function

The purpose of this section is to show that the cost function C(Q ,M) given in equation (1) is convex. The proof utilizes the standard definition for convexity. Furthermore, it should be noted that the sum of convex functions is a convex function. Therefore, it is sufficient to analyze only the following part of the total cost function:

$$C_s(Q,m_s) = Q \cdot \alpha_s \cdot D + \frac{Q \cdot \gamma_s \cdot D}{\left[(\delta_s)^{m_s} - 1\right]} + S_s \cdot \frac{D}{Q} + T_s \cdot m_s \cdot \frac{D}{Q} \qquad (2)$$

where:

$$\alpha_s = \frac{1}{2} \cdot \left(\frac{1}{\min(P)_{s,s+1}} - \frac{1}{\max(P)_{s,s+1}} \right) \cdot c_s \; ; \quad \gamma_s = \frac{c_s \cdot (\delta_s - 1)}{\max(P)_{s,s+1}} \quad \text{and} \quad \delta_s = \frac{\max(P)_{s,s+1}}{\min(P)_{s,s+1}}$$

The smallest batch size can be derived from the lot size Q by using relation (3).

$$q^s_{Min} = Q \cdot \frac{(\delta_s) - 1}{(\delta_s)^{m_s} - 1} \qquad \text{for } s = 1, 2, \ldots, S \qquad (3)$$

By solving equation (3) for m_s we obtain:

$$m_s = \frac{\ln\left[\frac{Q}{q^s_{Min}} \cdot (\delta_s - 1) + 1 \right]}{\ln[\delta_s]} \qquad (4)$$

Substituting equation (4) back into equation (2) and using relation (3) gives:

$$C_s(Q,q^s_{Min}) = Q\cdot\alpha_s\cdot D + \frac{q^s_{Min}\cdot c_s\cdot D}{\max(P)_{s,s+1}} + S_s\cdot\frac{D}{Q} + T_s\cdot\frac{\ln\left[\frac{Q}{q^s_{Min}}\cdot(\delta_s-1)+1\right]}{\ln[\delta_s]}\cdot\frac{D}{Q} \tag{5}$$

According to the standard definition of convexity, the function $C_s(Q, q^s_{Min})$ in equation (5) is convex if the following statement is true:

$$C_s\left[\lambda\cdot Q_1+(1-\lambda)\cdot Q_2;\lambda\cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}\right] \le \lambda\cdot C_s(Q_1;q^s_{Min,1})+(1-\lambda)\cdot C_s(Q_2;q^s_{Min,2}) \tag{6}$$

where λ is any real value such that: $0 \le \lambda \le 1$.

In the above equation (Q_1; $q^s_{Min,1}$) and (Q_2; $q^s_{Min,2}$) represent two distinct nonnegative pairs of the variables Q and q^s_{Min}. In order to continue with the proof we define now the relevant functions with respect to inequality (6).

$$C_s\left[\lambda\cdot Q_1+(1-\lambda)\cdot Q_2;\lambda\cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}\right] = \left[\lambda\cdot Q_1+(1-\lambda)\cdot Q_2\right]\cdot\alpha_s\cdot D + \left[\lambda\cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}\right]\cdot\frac{c_s\cdot D}{\max(P)_{s,s+1}} + S_s\cdot\frac{D}{\lambda\cdot Q_1+(1-\lambda)\cdot Q_2} +$$

$$+ T_s\cdot\frac{\ln\left[\frac{\lambda\cdot Q_1+(1-\lambda)\cdot Q_2}{\lambda\cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\cdot\frac{D}{\lambda\cdot Q_1+(1-\lambda)\cdot Q_2} \tag{7}$$

$$\lambda\cdot C_s\left[\cdot Q_1;q^s_{Min,1}\right] = \lambda\cdot Q_1\cdot\alpha_s\cdot D + \lambda\cdot q^s_{Min,1}\cdot\frac{c_s\cdot D}{\max(P)_{s,s+1}} + \lambda\cdot S_s\cdot\frac{D}{Q_1} + \lambda\cdot T_s\cdot\frac{\ln\left[\frac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\cdot\frac{D}{Q_1} \tag{8}$$

$$(1-\lambda)\cdot C_s\left[\cdot Q_2;q^s_{Min,2}\right] = (1-\lambda)\cdot Q_2\cdot\alpha_s\cdot D + (1-\lambda)\cdot q^s_{Min,2}\cdot\frac{c_s\cdot D}{\max(P)_{s,s+1}} + (1-\lambda)\cdot S_s\cdot\frac{D}{Q_2} + (1-\lambda)\cdot T_s\cdot\frac{\ln\left[\frac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\cdot\frac{D}{Q_2} \tag{9}$$

Inserting (7), (8) and (9) in inequality (6) and subtracting $[\lambda \cdot Q_1+(1-\lambda)\cdot Q_2]\cdot \alpha_s \cdot D$ as well as $[\lambda \cdot q^S_{Min,1} +(1-\lambda)\cdot q^S_{Min,2}]\cdot c_s \cdot D/\max(P)_{s,s+1}$ on both sides of the inequaltity results after simplification in:

$$\frac{S_s}{\lambda \cdot Q_1+(1-\lambda)\cdot Q_2}+T_s\cdot\frac{\ln\left[\frac{\lambda \cdot Q_1+(1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\cdot\frac{1}{\lambda \cdot Q_1+(1-\lambda)\cdot Q_2}\le\lambda\cdot S_s\cdot\frac{1}{Q_1}+\lambda\cdot T_s\cdot\frac{\ln\left[\frac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\cdot\frac{1}{Q_1}+(1-\lambda)\cdot S_s\cdot\frac{1}{Q_2}+(1-\lambda)\cdot T_s\cdot\frac{\ln\left[\frac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\cdot\frac{1}{Q_2} \qquad (10)$$

The next step is to multiply both sides of inequality (10) one after the other with Q_1, Q_2 and $[\lambda \cdot Q_1+(1-\lambda)\cdot Q_2]$. After doing this and rearraging the terms we obtain:

$$\frac{S_s\cdot Q_1}{[\lambda\cdot Q_1+(1-\lambda)Q_2]}+T_s\cdot\frac{\frac{\ln\left[\frac{\lambda \cdot Q_1+(1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\cdot Q_1}{[\lambda\cdot Q_1+(1-\lambda)Q_2]}\le\lambda\cdot S_s+(1-\lambda)\cdot S_s\cdot\frac{Q_1}{Q_2}+\lambda\cdot T_s\cdot\frac{\ln\left[\frac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}+(1-\lambda)\cdot T_s\cdot\frac{Q_1}{Q_2}\cdot\frac{\ln\left[\frac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}$$

$$\frac{S_s\cdot Q_1\cdot Q_2}{[\lambda\cdot Q_1+(1-\lambda)Q_2]}+T_s\cdot\frac{\frac{\ln\left[\frac{\lambda \cdot Q_1+(1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\cdot Q_1\cdot Q_2}{[\lambda\cdot Q_1+(1-\lambda)Q_2]}\le\lambda\cdot S_s\cdot Q_2+(1-\lambda)\cdot S_s\cdot Q_1+\lambda\cdot T_s\cdot Q_2\cdot\frac{\ln\left[\frac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}+(1-\lambda)\cdot T_s\cdot Q_1\cdot\frac{\ln\left[\frac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}$$

$$S_s\cdot Q_1\cdot Q_2+T_s\cdot\frac{\ln\left[\frac{\lambda \cdot Q_1+(1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\cdot Q_1\cdot Q_2\le[\lambda\cdot S_s\cdot Q_2+(1-\lambda)\cdot S_s\cdot Q_1]\cdot[\lambda\cdot Q_1+(1-\lambda)Q_2]+\left\{\lambda\cdot T_s\cdot Q_2\cdot\frac{\ln\left[\frac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}+(1-\lambda)\cdot T_s\cdot Q_1\cdot\frac{\ln\left[\frac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{\ln(\delta_s)}\right\}\cdot[\lambda\cdot Q_1+(1-\lambda)Q_2]$$

(11)

Upon the algebraic manipulations given below the statement in (11) reduces to:

$$S_s \cdot Q_1 \cdot Q_2 + T_s \cdot \frac{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda) \cdot Q_2}{\lambda \cdot q^s_{Min,1} + (1-\lambda) \cdot q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \cdot Q_1 \cdot Q_2 \le$$

$$\le (\lambda^2 \cdot Q_1 \cdot Q_2 + \lambda \cdot Q_2^2 - \lambda^2 \cdot Q_2^2) \cdot S_s + (\lambda \cdot Q_1^2 + Q_1 \cdot Q_2 - \lambda \cdot Q_1 \cdot Q_2 - \lambda^2 \cdot Q_1^2 - \lambda \cdot Q_1 \cdot Q_2 + \lambda^2 \cdot Q_1 \cdot Q_2) \cdot S_s + \left\{\lambda \cdot T_s \cdot Q_2 \cdot \frac{\ln\left[\frac{Q_1}{q^s_{Min,1}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + (1-\lambda) \cdot T_s \cdot Q_1 \cdot \frac{\ln\left[\frac{Q_2}{q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)}\right\} \cdot [\lambda \cdot Q_1 + (1-\lambda) Q_2]$$

$$T_s \cdot \frac{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda) \cdot Q_2}{\lambda \cdot q^s_{Min,1} + (1-\lambda) \cdot q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \cdot Q_1 \cdot Q_2 \le$$

$$\le (\lambda^2 \cdot Q_1 \cdot Q_2 + \lambda \cdot Q_2^2 - \lambda^2 \cdot Q_2^2) \cdot S_s + (\lambda \cdot Q_1^2 - \lambda \cdot Q_1 \cdot Q_2 - \lambda^2 \cdot Q_1^2 - \lambda \cdot Q_1 \cdot Q_2 + \lambda^2 \cdot Q_1 \cdot Q_2) \cdot S_s + \left\{\lambda \cdot T_s \cdot Q_2 \cdot \frac{\ln\left[\frac{Q_1}{q^s_{Min,1}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + (1-\lambda) \cdot T_s \cdot Q_1 \cdot \frac{\ln\left[\frac{Q_2}{q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)}\right\} \cdot [\lambda \cdot Q_1 + (1-\lambda) Q_2]$$

$$T_s \cdot \frac{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda) \cdot Q_2}{\lambda \cdot q^s_{Min,1} + (1-\lambda) \cdot q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \cdot Q_1 \cdot Q_2 \le (\lambda \cdot Q_1^2 - \lambda^2 \cdot Q_1^2 - 2 \cdot \lambda \cdot Q_1 \cdot Q_2 + 2 \cdot \lambda^2 \cdot Q_1 \cdot Q_2 + \lambda \cdot Q_2^2 - \lambda^2 \cdot Q_2^2) \cdot S_s + \left\{\lambda \cdot T_s \cdot Q_2 \cdot \frac{\ln\left[\frac{Q_1}{q^s_{Min,1}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + (1-\lambda) \cdot T_s \cdot Q_1 \cdot \frac{\ln\left[\frac{Q_2}{q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)}\right\} \cdot [\lambda \cdot Q_1 + (1-\lambda) Q_2]$$

$$T_s \cdot \frac{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1} + (1-\lambda)\cdot q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \cdot Q_1 \cdot Q_2 \le [\lambda \cdot (1-\lambda)\cdot Q_1^2 - 2 \cdot \lambda \cdot (1-\lambda) \cdot Q_1 \cdot Q_2 + \lambda(1-\lambda)\cdot Q_2^2] \cdot S_s + \left\{ \lambda \cdot T_s \cdot Q_2 \cdot \frac{\ln\left[\frac{Q_1}{q^s_{Min,1}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + (1-\lambda)\cdot T_s \cdot Q_1 \cdot \frac{\ln\left[\frac{Q_2}{q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \right\} \cdot [\lambda \cdot Q_1 + (1-\lambda) Q_2]$$

$$T_s \cdot \frac{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1} + (1-\lambda)\cdot q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \cdot Q_1 \cdot Q_2 \le \lambda \cdot (1-\lambda) \cdot (Q_1 - Q_2)^2 \cdot S_s + \left\{ \lambda \cdot T_s \cdot Q_2 \cdot \frac{\ln\left[\frac{Q_1}{q^s_{Min,1}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + (1-\lambda)\cdot T_s \cdot Q_1 \cdot \frac{\ln\left[\frac{Q_2}{q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \right\} \cdot [\lambda \cdot Q_1 + (1-\lambda) Q_2] \qquad |:Q_1$$

$$T_s \cdot \frac{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1} + (1-\lambda)\cdot q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \cdot Q_2 \le \left\{ \lambda \cdot T_s \cdot \frac{Q_2}{Q_1} \cdot \frac{\ln\left[\frac{Q_1}{q^s_{Min,1}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + (1-\lambda)\cdot T_s \cdot \frac{\ln\left[\frac{Q_2}{q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \right\} \cdot [\lambda \cdot Q_1 + (1-\lambda) Q_2] + \frac{\lambda \cdot (1-\lambda) \cdot (Q_1 - Q_2)^2 \cdot S_s}{Q_1} \qquad |:Q_2$$

$$T_s \cdot \frac{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1} + (1-\lambda)\cdot q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \le \left\{ \lambda \cdot T_s \cdot \frac{1}{Q_1} \cdot \frac{\ln\left[\frac{Q_1}{q^s_{Min,1}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + (1-\lambda)\cdot T_s \cdot \frac{1}{Q_2} \cdot \frac{\ln\left[\frac{Q_2}{q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \right\} \cdot [\lambda \cdot Q_1 + (1-\lambda) Q_2] + \frac{\lambda \cdot (1-\lambda) \cdot (Q_1 - Q_2)^2 \cdot S_s}{Q_1 \cdot Q_2} \qquad |:[\lambda \cdot Q_1 + (1-\lambda) \cdot Q_2]$$

$$T_s \cdot \frac{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1} + (1-\lambda)\cdot q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \cdot \frac{1}{[\lambda \cdot Q_1 + (1-\lambda) Q_2]} \le \lambda \cdot T_s \cdot \frac{1}{Q_1} \cdot \frac{\ln\left[\frac{Q_1}{q^s_{Min,1}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + (1-\lambda)\cdot T_s \cdot \frac{1}{Q_2} \cdot \frac{\ln\left[\frac{Q_2}{q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + \frac{\lambda \cdot (1-\lambda) \cdot (Q_1 - Q_2)^2 \cdot S_s}{Q_1 \cdot Q_2 \cdot [\lambda \cdot Q_1 + (1-\lambda) Q_2]} \qquad |:T_s$$

$$\frac{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda)\cdot Q_2}{\lambda \cdot q^s_{Min,1} + (1-\lambda)\cdot q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} \cdot \frac{1}{[\lambda \cdot Q_1 + (1-\lambda) Q_2]} \le \lambda \cdot \frac{1}{Q_1} \cdot \frac{\ln\left[\frac{Q_1}{q^s_{Min,1}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + (1-\lambda) \cdot \frac{1}{Q_2} \cdot \frac{\ln\left[\frac{Q_2}{q^s_{Min,2}} \cdot (\delta_s - 1) + 1\right]}{\ln(\delta_s)} + \frac{\lambda \cdot (1-\lambda) \cdot (Q_1 - Q_2)^2 \cdot S_s}{T_s \cdot Q_1 \cdot Q_2 \cdot [\lambda \cdot Q_1 + (1-\lambda) Q_2]} \qquad |\cdot \ln(\delta_s)$$

$$\ln\left[\frac{\lambda\cdot Q_1+(1-\lambda)\cdot Q_2}{\lambda\cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]\cdot\frac{1}{[\lambda\cdot Q_1+(1-\lambda)Q_2]}\le\lambda\cdot\ln\left[\frac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]\cdot\frac{1}{Q_1}+(1-\lambda)\cdot\ln\left[\frac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]\cdot\frac{1}{Q_2}+\frac{\lambda\cdot(1-\lambda)\cdot(Q_1-Q_2)^2\cdot S_s\cdot\ln(\delta_s)}{T_s\cdot[\lambda\cdot Q_1+(1-\lambda)Q_2]\cdot Q_1\cdot Q_2} \tag{12}$$

From an inspection of (12) it can be seen that the last term of this inequality is always nonnegative. Therefore, in order to prove the convexity of the total cost function we must show that the following inequality holds:

$$\lambda\cdot\frac{\ln\left[\frac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}{Q_1}+(1-\lambda)\cdot\frac{\ln\left[\frac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{Q_2}-\frac{\ln\left[\frac{\lambda\cdot Q_1+(1-\lambda)Q_2}{\lambda\cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}{[\lambda\cdot Q_1+(1-\lambda)Q_2]}+\frac{\lambda\cdot(1-\lambda)\cdot(Q_1-Q_2)^2\cdot S_s\cdot\ln(\delta_s)}{T_s\cdot[\lambda\cdot Q_1+(1-\lambda)Q_2]\cdot Q_1\cdot Q_2}\ge 0 \tag{13}$$

Inequality (13) can also be rewritten as follows:

$$\lambda\cdot\frac{\frac{1}{Q_1}}{\frac{1}{\ln\left[\frac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}}+(1-\lambda)\cdot\frac{\frac{1}{Q_2}}{\frac{1}{\ln\left[\frac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}}-\frac{\frac{1}{[\lambda\cdot Q_1+(1-\lambda)Q_2]}}{\frac{1}{\ln\left[\frac{\lambda\cdot Q_1+(1-\lambda)Q_2}{\lambda\cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}}+\frac{\lambda\cdot(1-\lambda)\cdot(Q_1-Q_2)^2\cdot S_s\cdot\ln(\delta_s)}{T_s\cdot[\lambda\cdot Q_1+(1-\lambda)Q_2]\cdot Q_1\cdot Q_2}\ge 0 \tag{14}$$

Furthermore, as shown in section 3 the adjacent inequality is always valid.

$$\frac{[\lambda\cdot Q_1+(1-\lambda)Q_2]}{\ln\left[\frac{\lambda\cdot Q_1+(1-\lambda)Q_2}{\lambda\cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}\ge\lambda\cdot\frac{Q_1}{\ln\left[\frac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}+(1-\lambda)\cdot\frac{Q_2}{\ln\left[\frac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]} \tag{15}$$

From the inequality given above it can be concluded that also the next statement is always true.

$$\frac{1}{\dfrac{[\lambda\cdot Q_1+(1-\lambda)Q_2]}{\ln\left[\dfrac{\lambda\cdot Q_1+(1-\lambda)Q_2}{\lambda\cdot q^s_{Min,1}+(1-\lambda)\cdot q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}}\le\frac{1}{\lambda\cdot\dfrac{Q_1}{\ln\left[\dfrac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}+(1-\lambda)\cdot\dfrac{Q_2}{\ln\left[\dfrac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}} \tag{16}$$

Using the result given in (16) we can deduce that if we can prove the inequality stated below then it is shown that inequality (14) is always greater than or equal zero.

$$\lambda\cdot\frac{1}{\dfrac{Q_1}{\ln\left[\dfrac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}}+(1-\lambda)\cdot\frac{1}{\dfrac{Q_2}{\ln\left[\dfrac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}}-\frac{1}{\lambda\cdot\dfrac{Q_1}{\ln\left[\dfrac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}+(1-\lambda)\cdot\dfrac{Q_2}{\ln\left[\dfrac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}}+\frac{\lambda\cdot(1-\lambda)\cdot(Q_1-Q_2)^2\cdot S_s\cdot\ln(\delta_s)}{T_s\cdot[\lambda\cdot Q_1+(1-\lambda)Q_2]\cdot Q_1\cdot Q_2}\ge 0 \tag{17}$$

Again, after the algebraic manipulations shown subsequent the statement in (17) results in inequality (18). In order to facilitate the understanding of our calculations we use the following simplifications:

$$\omega_1=\frac{Q_1}{\ln\left[\dfrac{Q_1}{q^s_{Min,1}}\cdot(\delta_s-1)+1\right]}\qquad \omega_2=\frac{Q_2}{\ln\left[\dfrac{Q_2}{q^s_{Min,2}}\cdot(\delta_s-1)+1\right]}\qquad \omega_3=\frac{\lambda\cdot(1-\lambda)\cdot(Q_1-Q_2)^2\cdot S_s\cdot\ln(\delta_s)}{T_s\cdot[\lambda\cdot Q_1+(1-\lambda)Q_2]\cdot Q_1\cdot Q_2}$$

Inserting the terms ω_1, ω_2 and ω_3 in inequality (17) gives:

$$\frac{1}{\lambda\cdot\omega_1+(1-\lambda)\cdot\omega_2}\le\lambda\cdot\frac{1}{\omega_1}+(1-\lambda)\cdot\frac{1}{\omega_2}+\omega_3 \qquad |\cdot\omega_1$$

$$\frac{\omega_1}{\lambda\cdot\omega_1+(1-\lambda)\cdot\omega_2}\le\lambda+(1-\lambda)\cdot\frac{\omega_1}{\omega_2}+\omega_3\cdot\omega_1 \qquad |\cdot\omega_2$$

$$\frac{\omega_1 \cdot \omega_2}{\lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2} \leq \lambda \cdot \omega_2 + (1-\lambda) \cdot \omega_1 + \omega_3 \cdot \omega_1 \cdot \omega_2 \qquad \Big| \cdot \lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2$$

$$\omega_1 \cdot \omega_2 \leq \lambda \cdot \omega_2 \cdot [\lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2] + (1-\lambda) \cdot \omega_1 \cdot [\lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2] + \omega_3 \cdot \omega_1 \cdot \omega_2 \cdot [\lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2]$$

$$\omega_1 \cdot \omega_2 \leq \lambda^2 \cdot \omega_1 \cdot \omega_2 + \lambda \cdot \omega_2^2 - \lambda^2 \cdot \omega_2^2 + \lambda \cdot \omega_1^2 + \omega_1 \cdot \omega_2 - \lambda \cdot \omega_1 \cdot \omega_2 - \lambda^2 \cdot \omega_1^2 - \lambda \cdot \omega_1 \cdot \omega_2 + \lambda^2 \cdot \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_1 \cdot \omega_2 \cdot [\lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2]$$

$$0 \leq \lambda^2 \cdot \omega_1 \cdot \omega_2 + \lambda \cdot \omega_2^2 - \lambda^2 \cdot \omega_2^2 + \lambda \cdot \omega_1^2 - \lambda \cdot \omega_1 \cdot \omega_2 - \lambda^2 \cdot \omega_1^2 - \lambda \cdot \omega_1 \cdot \omega_2 + \lambda^2 \cdot \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_1 \cdot \omega_2 \cdot [\lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2]$$

$$0 \leq \lambda \cdot \omega_1^2 - \lambda^2 \cdot \omega_1^2 + \lambda \cdot \omega_2^2 - \lambda^2 \cdot \omega_2^2 - 2 \cdot \lambda \cdot \omega_1 \cdot \omega_2 + 2 \cdot \lambda^2 \cdot \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_1 \cdot \omega_2 \cdot [\lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2]$$

$$0 \leq \lambda \cdot (1-\lambda) \cdot \omega_1^2 + \lambda \cdot (1-\lambda) \cdot \omega_2^2 - 2 \cdot \lambda \cdot (1-\lambda) \cdot \omega_1 \cdot \omega_2 + \omega_3 \cdot \omega_1 \cdot \omega_2 \cdot [\lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2]$$

$$0 \leq \lambda \cdot (1-\lambda) \cdot (\omega_1 - \omega_2)^2 + \omega_3 \cdot \omega_1 \cdot \omega_2 \cdot [\lambda \cdot \omega_1 + (1-\lambda) \cdot \omega_2]$$

Countermand the simplifications results in:

$$0 \leq \lambda \cdot (1-\lambda) \cdot \left\{ \frac{Q1}{\ln\left[\frac{Q1}{q^S_{Min,1}} \cdot (\delta_S - 1) + 1\right]} - \frac{Q2}{\ln\left[\frac{Q2}{q^S_{Min,2}} \cdot (\delta_S - 1) + 1\right]} \right\}^2 + \frac{\lambda \cdot (1-\lambda) \cdot (Q1 - Q2)^2 \cdot S_S \cdot \ln(\delta_S)}{T_S \cdot [\lambda \cdot Q1 + (1-\lambda) Q2] \cdot Q1 \cdot Q2} \cdot \frac{Q1}{\ln\left[\frac{Q1}{q^S_{Min,1}} \cdot (\delta_S - 1) + 1\right]} \cdot \frac{Q2}{\ln\left[\frac{Q2}{q^S_{Min,2}} \cdot (\delta_S - 1) + 1\right]} \cdot \left\{ \lambda \cdot \frac{Q1}{\ln\left[\frac{Q1}{q^S_{Min,1}} \cdot (\delta_S - 1) + 1\right]} + (1-\lambda) \cdot \frac{Q2}{\ln\left[\frac{Q2}{q^S_{Min,2}} \cdot (\delta_S - 1) + 1\right]} \right\} \tag{18}$$

Since all terms on the right hand side of inequality (18) are individual nonnegative the statement in (18) is valid. Therefore, the total cost function C(Q, M) given in equation (1) is convex.

3 Additional Analytical Results

The aim of this section is to verify the validity of inequality (15) given in section 2. Inequality (15) was defined as follows:

$$\frac{[\lambda \cdot Q_1 + (1-\lambda)Q_2]}{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda)Q_2}{\lambda \cdot q^S_{Min,1} + (1-\lambda)\cdot q^S_{Min,2}} \cdot (\delta_S - 1) + 1\right]} \geq \lambda \cdot \frac{Q_1}{\ln\left[\frac{Q_1}{q^S_{Min,1}} \cdot (\delta_S - 1) + 1\right]} + (1-\lambda) \cdot \frac{Q_2}{\ln\left[\frac{Q_2}{q^S_{Min,2}} \cdot (\delta_S - 1) + 1\right]} \qquad (15)$$

From an closer examination of inequality (15) it can be seen that the basis for all individual terms is given by the function:

$$f(Q, q^S_{Min}) = \frac{Q}{\ln\left[\frac{Q}{q^S_{Min}} \cdot (\delta_S - 1) + 1\right]} \qquad (19)$$

The remainder of this section is organized as follows. At first we analyze the course of the function f(Q, q^S_{Min}) for a given value of q^S_{Min}. In the next subsection we examine the course of the function f(Q, q^S_{Min}) for a given value of Q. In the third and last subsection we combine the results obtained in the previous two subsections in order to prove the validity of inequality (15).

3.1 Analysis of f(Q, q^S_{Min}) for a given value of q^S_{Min}

For a given value of q^S_{Min} the expression for f(Q, q^S_{Min}) can be simplified as follows:

$$f(Q) = \frac{Q}{\ln(a \cdot Q + 1)} \qquad \text{where: } a = (\delta_S - 1)/q^S_{Min} \qquad (20)$$

By differentiating (20) with respect to Q we obtain the equation stated below.

$$\frac{df(Q)}{dQ}=\frac{\ln(a\cdot Q+1)-Q\cdot\frac{a}{a\cdot Q+1}}{[\ln(a\cdot Q+1)]^2} \tag{21}$$

With z = a·Q+1 we can express the term ln(z) by using the following power series:

$$\ln(z)=\frac{z-1}{z}+\frac{(z-1)^2}{2\cdot z^2}+\frac{(z-1)^3}{3\cdot z^3}+\ldots+\frac{(z-1)^n}{n\cdot z^n}+\ldots \tag{22}$$

Using z = a·Q+1 (22) can be rewritten as follows:

$$\ln(a\cdot Q+1)=\frac{a\cdot Q}{a\cdot Q+1}+\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{3\cdot(a\cdot Q+1)^3}+\ldots+\frac{(a\cdot Q)^n}{n\cdot(a\cdot Q+1)^n}+\ldots \tag{23}$$

Now we substitute (23) back into equation (21) and the first derivation becomes:

$$\frac{df(Q)}{dQ}=\frac{\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{3\cdot(a\cdot Q+1)^3}+\ldots+\frac{(a\cdot Q)^n}{n\cdot(a\cdot Q+1)^n}+\ldots}{\left[\frac{a\cdot Q}{a\cdot Q+1}+\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{3\cdot(a\cdot Q+1)^3}+\ldots+\frac{(a\cdot Q)^n}{n\cdot(a\cdot Q+1)^n}+\ldots\right]^2} \tag{24}$$

From an inspection of (24) it can be seen that df(Q)/dQ is always nonnegative. Therefore, we can conclude that the function f(Q) is a *monotonous increasing function. We continue our analysis with an closer examination of the second derivation of f(Q) with respect to Q.* Note, that df(Q)/dQ (equation (21)) can be rewritten as follows:

$$\frac{df(Q)}{dQ}=\frac{1}{\ln(a\cdot Q+1)}-\frac{\frac{a\cdot Q}{a\cdot Q+1}}{[\ln(a\cdot Q+1]^2} \tag{25}$$

$$\frac{d^2f(Q)}{dQ^2}=-\frac{\frac{a}{a\cdot Q+1}}{[\ln(a\cdot Q+1)]^2}-\frac{\left\{[\ln(a\cdot Q+1)]^2\cdot\left[\frac{(a\cdot Q+1)\cdot a-a\cdot Q\cdot a}{(a\cdot Q+1)^2}\right]-\frac{a\cdot Q}{a\cdot Q+1}\cdot 2\cdot\ln(a\cdot Q+1)\cdot\frac{a}{a\cdot Q+1}\right\}}{[\ln(a\cdot Q+1)]^4} \tag{26}$$

After algebraic manipulations and rearranging the terms (26) can be stated as follows:

$$\frac{d^2f(Q)}{dQ^2}=-\frac{\frac{a}{a\cdot Q+1}}{[\ln(a\cdot Q+1)]^2}-\frac{\left\{[\ln(a\cdot Q+1)]\cdot\frac{a}{(a\cdot Q+1)^2}-\frac{a\cdot Q}{a\cdot Q+1}\cdot 2\cdot\frac{a}{a\cdot Q+1}\right\}}{[\ln(a\cdot Q+1)]^3}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{-\frac{a}{a\cdot Q+1}\cdot\ln(a\cdot Q+1)-\left\{[\ln(a\cdot Q+1)]\cdot\frac{a}{(a\cdot Q+1)^2}-\frac{a\cdot Q}{a\cdot Q+1}\cdot 2\cdot\frac{a}{a\cdot Q+1}\right\}}{[\ln(a\cdot Q+1)]^3}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{-\frac{a}{a\cdot Q+1}\cdot\ln(a\cdot Q+1)-\frac{a}{a\cdot Q+1}\cdot\left\{[\ln(a\cdot Q+1)]\cdot\frac{1}{a\cdot Q+1}-2\cdot\frac{a\cdot Q}{a\cdot Q+1}\right\}}{[\ln(a\cdot Q+1)]^3}\qquad\left|\cdot\frac{\frac{a\cdot Q+1}{a}}{\frac{a\cdot Q+1}{a}}\right.$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{\left\{-[\ln(a\cdot Q+1)]-\left[\ln(a\cdot Q+1)\cdot\frac{1}{a\cdot Q+1}-2\cdot\frac{a\cdot Q}{a\cdot Q+1}\right]\right\}}{\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}\qquad(27)$$

By using the power series given in (23) the equation (27) can be expressed in the following way:

$$\frac{d^2f(Q)}{dQ^2}=\frac{-\left[\frac{a\cdot Q}{a\cdot Q+1}+\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{3\cdot(a\cdot Q+1)^3}+\frac{(a\cdot Q)^4}{4\cdot(a\cdot Q+1)^4}+\frac{(a\cdot Q)^5}{5\cdot(a\cdot Q+1)^5}+\ldots+\frac{(a\cdot Q)^n}{n\cdot(a\cdot Q+1)^n}+\ldots\right]-\left[\frac{a\cdot Q}{a\cdot Q+1}+\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{3\cdot(a\cdot Q+1)^3}+\frac{(a\cdot Q)^4}{4\cdot(a\cdot Q+1)^4}+\frac{(a\cdot Q)^5}{5\cdot(a\cdot Q+1)^5}+\ldots+\frac{(a\cdot Q)^n}{n\cdot(a\cdot Q+1)^n}+\ldots\right]\cdot\frac{1}{a\cdot Q+1}+2\cdot\frac{a\cdot Q}{a\cdot Q+1}}{\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{\frac{a\cdot Q}{a\cdot Q+1}-\left[\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{3\cdot(a\cdot Q+1)^3}+\frac{(a\cdot Q)^4}{4\cdot(a\cdot Q+1)^4}+\frac{(a\cdot Q)^5}{5\cdot(a\cdot Q+1)^5}+\ldots+\frac{(a\cdot Q)^n}{n\cdot(a\cdot Q+1)^n}+\ldots\right]-\left[\frac{a\cdot Q}{a\cdot Q+1}+\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{3\cdot(a\cdot Q+1)^3}+\frac{(a\cdot Q)^4}{4\cdot(a\cdot Q+1)^4}+\frac{(a\cdot Q)^5}{5\cdot(a\cdot Q+1)^5}+\ldots+\frac{(a\cdot Q)^n}{n\cdot(a\cdot Q+1)^n}+\ldots\right]\cdot\frac{1}{a\cdot Q+1}}{\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{\frac{a\cdot Q}{a\cdot Q+1}-\frac{a\cdot Q}{a\cdot Q+1}\cdot\frac{1}{a\cdot Q+1}-\left[\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{3\cdot(a\cdot Q+1)^3}+\frac{(a\cdot Q)^4}{4\cdot(a\cdot Q+1)^4}+\frac{(a\cdot Q)^5}{5\cdot(a\cdot Q+1)^5}+\ldots+\frac{(a\cdot Q)^n}{n\cdot(a\cdot Q+1)^n}+\ldots\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}\qquad(28)$$

Equation (28) can also be stated as follows:

$$\frac{d^2f(Q)}{dQ^2}=\frac{\frac{a\cdot Q}{a\cdot Q+1}\cdot\left(1-\frac{1}{a\cdot Q+1}\right)-\frac{a\cdot Q}{a\cdot Q+1}\cdot\left[\frac{(a\cdot Q)^1}{2\cdot(a\cdot Q+1)^1}+\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^3}+\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^4}+...+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-1}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3} \tag{29}$$

Equation (29) reduces upon the following lines of algebra to:

$$\frac{d^2f(Q)}{dQ^2}=\frac{\frac{a\cdot Q}{a\cdot Q+1}\cdot\left(1-\frac{1}{a\cdot Q+1}\right)-\frac{a\cdot Q}{a\cdot Q+1}\cdot\left[\frac{(a\cdot Q)^1}{2\cdot(a\cdot Q+1)^1}+\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^3}+\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^4}+...+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-1}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}\qquad\Bigg|\cdot\frac{\frac{a\cdot Q+1}{a\cdot Q}}{\frac{a\cdot Q+1}{a\cdot Q}}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{\left(1-\frac{1}{a\cdot Q+1}\right)-\left[\frac{(a\cdot Q)^1}{2\cdot(a\cdot Q+1)^1}+\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^3}+\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^4}+...+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-1}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{\frac{a\cdot Q+1}{a\cdot Q}\cdot\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{\left(1-\frac{1}{a\cdot Q+1}\right)-\frac{1}{a\cdot Q+1}\cdot\left[\frac{a\cdot Q}{2}+\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^1}+\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^3}....+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-2}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{\frac{a\cdot Q+1}{a\cdot Q}\cdot\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}\qquad\Bigg|\cdot\frac{a\cdot Q+1}{a\cdot Q+1}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{(a\cdot Q+1)\cdot\left(1-\frac{1}{a\cdot Q+1}\right)-\left[\frac{a\cdot Q}{2}+\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^1}+\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^3}....+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-2}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{(a\cdot Q+1)\cdot\frac{a\cdot Q+1}{a\cdot Q}\cdot\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{a\cdot Q-\left[\frac{a\cdot Q}{2}+\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^1}+\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^3}....+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-2}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{(a\cdot Q+1)\cdot\frac{a\cdot Q+1}{a\cdot Q}\cdot\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}$$

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auxiliary calculation:

$$a\cdot Q-\frac{a\cdot Q}{2}\cdot\left(1+\frac{1}{a\cdot Q+1}\right)=a\cdot Q-\frac{a\cdot Q}{2}-\frac{a\cdot Q}{2\cdot(a\cdot Q+1)}=\frac{a\cdot Q}{2}-\frac{a\cdot Q}{2\cdot(a\cdot Q+1)}=\frac{a\cdot Q}{2}\cdot\left(1-\frac{1}{a\cdot Q+1}\right)=\frac{a\cdot Q}{2}\cdot\left(\frac{a\cdot Q+1}{a\cdot Q+1}-\frac{1}{a\cdot Q+1}\right)=\frac{a\cdot Q}{2}\cdot\frac{a\cdot Q}{a\cdot Q+1}=\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)}-\left[\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^1}+\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^3}....+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-2}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{(a\cdot Q+1)\cdot\frac{a\cdot Q+1}{a\cdot Q}\cdot\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}$$

auxiliary calculation:

$$\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)}-\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)}\cdot\left(1+\frac{1}{a\cdot Q+1}\right)=\frac{(a\cdot Q)^2}{2\cdot(a\cdot Q+1)}-\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)}-\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^2}=\frac{3\cdot(a\cdot Q)^2-2\cdot(a\cdot Q)^2}{6\cdot(a\cdot Q+1)}-\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^2}=\frac{(a\cdot Q)^2}{6\cdot(a\cdot Q+1)}-\frac{(a\cdot Q)^2}{3\cdot(a\cdot Q+1)^2}=\frac{(a\cdot Q)^2\cdot(a\cdot Q+1)-2\cdot(a\cdot Q)^2}{6\cdot(a\cdot Q+1)^2}=\frac{(a\cdot Q)^3+(a\cdot Q)^2-2\cdot(a\cdot Q)^2}{6\cdot(a\cdot Q+1)^2}$$

$$=\frac{(a\cdot Q)^3-(a\cdot Q)^2}{6\cdot(a\cdot Q+1)^2}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{\frac{(a\cdot Q)^3}{6\cdot(a\cdot Q+1)^2}-\frac{(a\cdot Q)^2}{6\cdot(a\cdot Q+1)^2}-\left[\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^2}+\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^3}....+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-2}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{(a\cdot Q+1)\cdot\frac{a\cdot Q+1}{a\cdot Q}\cdot\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{\frac{(a\cdot Q)^3}{6\cdot(a\cdot Q+1)^2}-\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^2}\cdot\left(1+\frac{1}{a\cdot Q+1}\right)-\frac{(a\cdot Q)^2}{6\cdot(a\cdot Q+1)^2}-\left[\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^3}....+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-2}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{(a\cdot Q+1)\cdot\frac{a\cdot Q+1}{a\cdot Q}\cdot\frac{(a\cdot Q+1)}{a}\cdot[\ln(a\cdot Q+1)]^3}$$

$$\frac{d^2f(Q)}{dQ^2}=\frac{\frac{(a\cdot Q)^3}{(a\cdot Q+1)^2}\cdot\left(\frac{1}{6}-\frac{1}{4}\right)-\frac{(a\cdot Q)^3}{4\cdot(a\cdot Q+1)^3}-\frac{(a\cdot Q)^2}{6\cdot(a\cdot Q+1)^2}-\left[\frac{(a\cdot Q)^4}{5\cdot(a\cdot Q+1)^3}+\frac{(a\cdot Q)^5}{6\cdot(a\cdot Q+1)^4}+...+\frac{(a\cdot Q)^{n-1}}{n\cdot(a\cdot Q+1)^{n-2}}+...\right]\cdot\left(1+\frac{1}{a\cdot Q+1}\right)}{\frac{(a\cdot Q+1)^3}{a\cdot a\cdot Q}\cdot[\ln(a\cdot Q+1)]^3}\qquad(30)$$

Since the nominator (denominator) in (30) is always negative (nonnegative) we can deduce that the second derivation of f(Q) with respect to Q is always negative. Using this result we can now conclude that the increase of f(Q) in direction of ascending values of Q gets smaller. The course of f(Q) is depicted in figure 1.

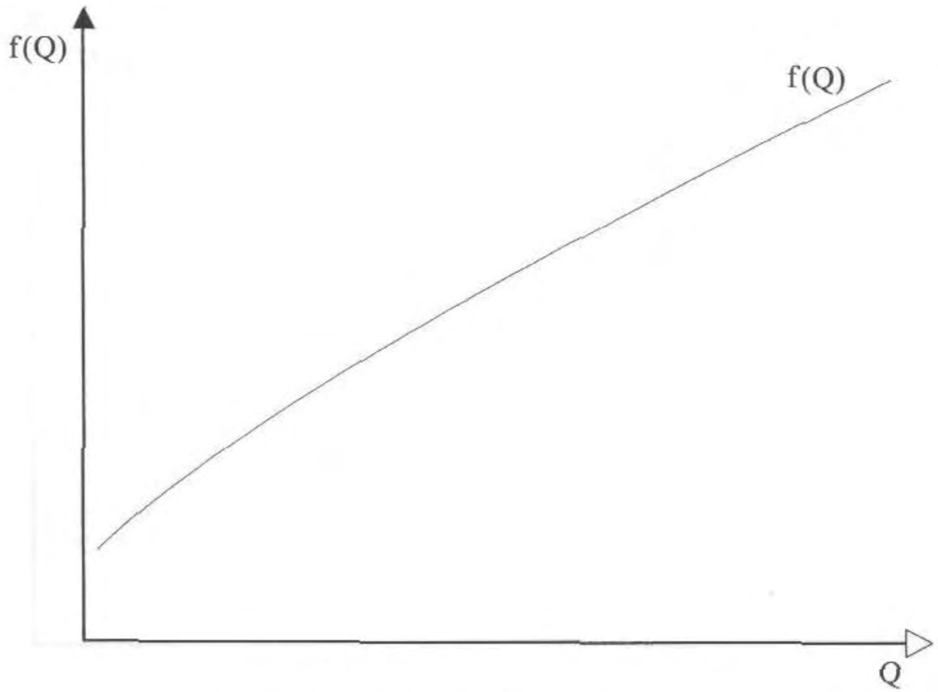

figure 1: Course of the function f(Q)

In order to generalize our results from the previous analysis the figure 2 shows the course of the function $f(Q, q^S_{Min})$ for three different given values of q^S_{Min} denoted by $q^S_{Min,1}$; $q^S_{Min,2}$ and $q^S_{Min,3}$ with $q^S_{Min,1} < q^S_{Min,2} < q^S_{Min,3}$. Recall $f(Q, q^S_{Min})$ was given by:

$$f(Q,q^S_{Min}) = \frac{Q}{\ln\left[\frac{Q}{q^S_{Min}}\cdot(\delta_S - 1) + 1\right]} \tag{19}$$

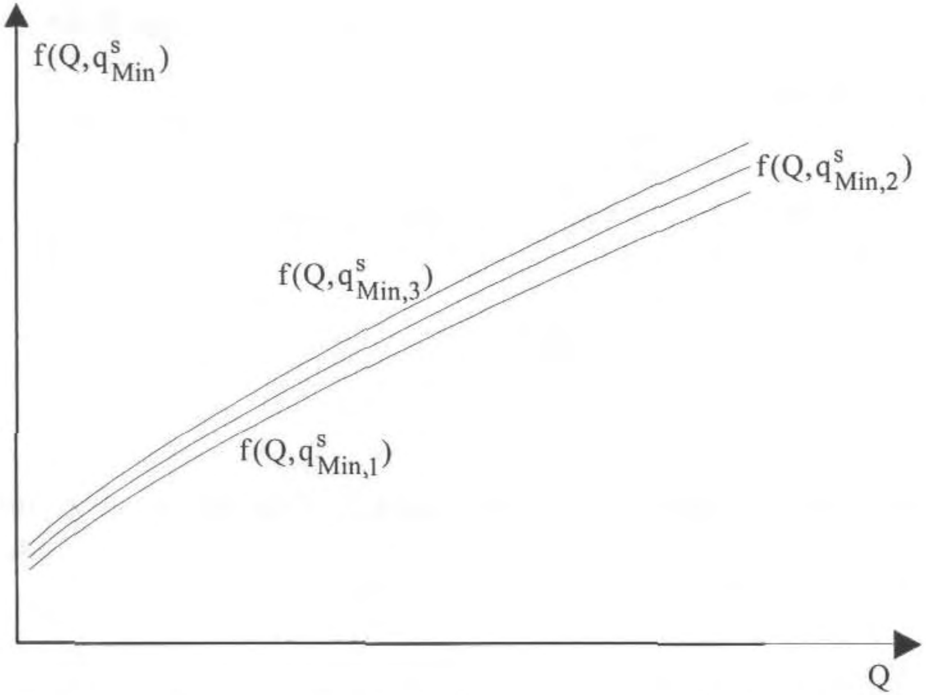

figure2: Course of the function $f(Q, q^S_{Min})$ for three different values of q^S_{Min}

3.2 Analysis of $f(Q, q^S_{Min})$ for a given value of Q

For a given value of Q the expression for $f(Q, q^S_{Min})$ can be simplified as follows:

$$f(q^s_{Min}) = \frac{a}{\ln\left(\frac{b}{q^s_{Min}} + 1\right)} \quad \text{where: } a = Q \text{ and } b = a\cdot(\delta_S - 1) \tag{31}$$

The first derivation of $f(q^S_{Min})$ with respect to q^S_{Min} is given by:

$$\frac{df(q^s_{Min})}{dq^s_{Min}} = \frac{-a \cdot \left[\frac{\frac{-b}{\left(q^s_{Min}\right)^2}}{\frac{b}{q^s_{Min}}+1}\right]}{\left[\ln\left(\frac{b}{q^s_{Min}}+1\right)\right]^2} = \frac{\frac{a \cdot b}{\left[b \cdot q^s_{Min} + \left(q^s_{Min}\right)^2\right]}}{\left[\ln\left(\frac{b}{q^s_{Min}}+1\right)\right]^2} \tag{32}$$

It can be seen that $df(q^S_{Min}) / dq^S_{Min}$ is always nonegative. Hence, the function $f(q^S_{Min})$ is a monotonous increasing function. Once again, we continue our analysis with examining the second derivation of $f(q^S_{Min})$ with respect to q^S_{Min}. The second derivation of $f(q^S_{Min})$ with respect to q^S_{Min} can be stated in the following way:

$$\frac{d^2 f\left(q^s_{Min}\right)}{d\left(q^s_{Min}\right)^2} = a \cdot b \cdot \frac{\left\{\left[\ln\left(\frac{b}{q^s_{Min}}+1\right)\right]^2 \cdot (-1) \cdot \frac{b + 2 \cdot q^s_{Min}}{\left[b \cdot q^s_{Min} + \left(q^s_{Min}\right)^2\right]^2} - \frac{1}{\left[b \cdot q^s_{Min} + \left(q^s_{Min}\right)^2\right]} \cdot 2 \cdot \ln\left(\frac{b}{q^s_{Min}}+1\right) \cdot (-1) \cdot \frac{b}{\left[b \cdot q^s_{Min} + \left(q^s_{Min}\right)^2\right]}\right\}}{\left[\ln\left(\frac{b}{q^s_{Min}}+1\right)\right]^4} \tag{33}$$

After simplification equation (33) can be written as follows:

$$\frac{d^2 f\left(q^s_{Min}\right)}{d\left(q^s_{Min}\right)^2} = \frac{a \cdot b}{\left[b \cdot q^s_{Min} + \left(q^s_{Min}\right)^2\right]^2} \cdot \frac{\left[-\ln\left(\frac{b}{q^s_{Min}} + 1\right) \cdot \left(b + 2 \cdot q^s_{Min}\right) + 2 \cdot b\right]}{\left[\ln\left(\frac{b}{q^s_{Min}} + 1\right)\right]^3}$$

$$\frac{d^2 f\left(q^s_{Min}\right)}{d\left(q^s_{Min}\right)^2} = \frac{-a \cdot b^2}{\left[b \cdot q^s_{Min} + \left(q^s_{Min}\right)^2\right]^2} \cdot \frac{\left[\ln\left(\frac{b}{q^s_{Min}} + 1\right) \cdot \left(1 + 2 \cdot \frac{q^s_{Min}}{b}\right) - 2\right]}{\left[\ln\left(\frac{b}{q^s_{Min}} + 1\right)\right]^3} \tag{34}$$

With $z = b/q^S_{Min} + 1$ we can express the term ln(z) by using the following power series:

$$\ln(z) = 2 \cdot \left[\frac{z-1}{z+1} + \frac{(z-1)^3}{3 \cdot (z+1)^3} + \frac{(z-1)^5}{5 \cdot (z+1)^5} + \ldots + \frac{(z-1)^{2n+1}}{(2n+1) \cdot (z+1)^{2n+1}} + \ldots\right] \tag{35}$$

After simplification (35) is given by:

$$\ln\left(\frac{b}{q^s_{Min}} + 1\right) = 2 \cdot \left[\frac{\frac{b}{q^s_{Min}}}{\frac{b}{q^s_{Min}} + 2} + \frac{\left(\frac{b}{q^s_{Min}}\right)^3}{3 \cdot \left(\frac{b}{q^s_{Min}} + 2\right)^3} + \frac{\left(\frac{b}{q^s_{Min}}\right)^5}{5 \cdot \left(\frac{b}{q^s_{Min}} + 2\right)^5} + \ldots + \frac{\left(\frac{b}{q^s_{Min}}\right)^{2n+1}}{(2n+1) \cdot \left(\frac{b}{q^s_{Min}} + 2\right)^{2n+1}} + \ldots\right] \tag{36}$$

Using the result given in equation (36) we can rewrite equation (34) as follows:

$$\frac{d^2 f\left(q^s_{Min}\right)}{d\left(q^s_{Min}\right)^2} = \frac{-a \cdot b^2}{\left[b \cdot q^s_{Min} + \left(q^s_{Min}\right)^2\right]^2} \cdot \frac{\left\{2 \cdot \left[\frac{\frac{b}{q^s_{Min}}}{\frac{b}{q^s_{Min}} + 2} + \frac{\left(\frac{b}{q^s_{Min}}\right)^3}{3 \cdot \left(\frac{b}{q^s_{Min}} + 2\right)^3} + \ldots + \frac{\left(\frac{b}{q^s_{Min}}\right)^{2n+1}}{(2n+1) \cdot \left(\frac{b}{q^s_{Min}} + 2\right)^{2n+1}} + \ldots\right] \cdot \left(1 + 2 \cdot \frac{q^s_{Min}}{b}\right) - 2\right\}}{\left[\ln\left(\frac{b}{q^s_{Min}} + 1\right)\right]^3}$$

$$\frac{d^2 f\left(q^s_{Min}\right)}{d\left(q^s_{Min}\right)^2} = \frac{-2 \cdot a \cdot b^2}{\left[b \cdot q^s_{Min} + \left(q^s_{Min}\right)^2\right]^2} \cdot \frac{\left\{\left[\frac{\left(\frac{b}{q^s_{Min}}\right)^3}{3 \cdot \left(\frac{b}{q^s_{Min}} + 2\right)^3} + \ldots + \frac{\left(\frac{b}{q^s_{Min}}\right)^{2n+1}}{(2n+1) \cdot \left(\frac{b}{q^s_{Min}} + 2\right)^{2n+1}} + \ldots\right] \cdot \left(1 + 2 \cdot \frac{q^s_{Min}}{b}\right) + \frac{\frac{b}{q^s_{Min}}}{\frac{b}{q^s_{Min}} + 2} \cdot \left(1 + 2 \cdot \frac{q^s_{Min}}{b}\right) - 1\right\}}{\left[\ln\left(\frac{b}{q^s_{Min}} + 1\right)\right]^3} \tag{37}$$

After further simplification equation (37) reduces to:

$$\frac{d^2 f\left(q^s_{Min}\right)}{d\left(q^s_{Min}\right)^2}=\frac{-2\cdot a\cdot b^2}{\left[b\cdot q^s_{Min}+\left(q^s_{Min}\right)^2\right]^2}\cdot\frac{\left\{\left[\frac{\left(\frac{b}{q^s_{Min}}\right)^3}{3\cdot\left(\frac{b}{q^s_{Min}}+2\right)^3}+\ldots+\frac{\left(\frac{b}{q^s_{Min}}\right)^{2n+1}}{(2n+1)\cdot\left(\frac{b}{q^s_{Min}}+2\right)^{2n+1}}+\ldots\right]\cdot\left(l+2\cdot\frac{q^s_{Min}}{b}\right)+\frac{\frac{b}{q^s_{Min}}}{\frac{b}{q^s_{Min}}+2}\cdot\left(l+2\cdot\frac{q^s_{Min}}{b}\right)-\frac{\frac{b}{q^s_{Min}}+2}{\frac{b}{q^s_{Min}}+2}\right\}}{\left[\ln\left(\frac{b}{q^s_{Min}}+1\right)\right]^3}$$

$$\frac{d^2 f\left(q^s_{Min}\right)}{d\left(q^s_{Min}\right)^2}=\frac{-2\cdot a\cdot b^2}{\left[b\cdot q^s_{Min}+\left(q^s_{Min}\right)^2\right]^2}\cdot\frac{\left\{\left[\frac{\left(\frac{b}{q^s_{Min}}\right)^3}{3\cdot\left(\frac{b}{q^s_{Min}}+2\right)^3}+\ldots+\frac{\left(\frac{b}{q^s_{Min}}\right)^{2n+1}}{(2n+1)\cdot\left(\frac{b}{q^s_{Min}}+2\right)^{2n+1}}+\ldots\right]\cdot\left(1+2\cdot\frac{q^s_{Min}}{b}\right)+\frac{\frac{b}{q^s_{Min}}+2}{\frac{b}{q^s_{Min}}+2}-\frac{\frac{b}{q^s_{Min}}+2}{\frac{b}{q^s_{Min}}+2}\right\}}{\left[\ln\left(\frac{b}{q^s_{Min}}+1\right)\right]^3}$$

$$\frac{d^2 f\left(q^s_{Min}\right)}{d\left(q^s_{Min}\right)^2}=\frac{-2\cdot a\cdot b^2}{\left[b\cdot q^s_{Min}+\left(q^s_{Min}\right)^2\right]^2}\cdot\frac{\left[\frac{\left(\frac{b}{q^s_{Min}}\right)^3}{3\cdot\left(\frac{b}{q^s_{Min}}+2\right)^3}+\ldots+\frac{\left(\frac{b}{q^s_{Min}}\right)^{2n+1}}{(2n+1)\cdot\left(\frac{b}{q^s_{Min}}+2\right)^{2n+1}}+\ldots\right]\cdot\left(1+2\cdot\frac{q^s_{Min}}{b}\right)}{\left[\ln\left(\frac{b}{q^s_{Min}}+1\right)\right]^3} \quad (38)$$

It can be seen that the first term of (38) is always negative whereas the second term is always nonnegative. Therefore, the second derivation of $f(q^S_{Min})$ with respect to q^S_{Min} is always negative. This shows us again that the increase of $f(q^S_{Min})$ in direction of ascending values of q^S_{Min} gets smaller. The course of $f(q^S_{Min})$ is depicted in figure 3.

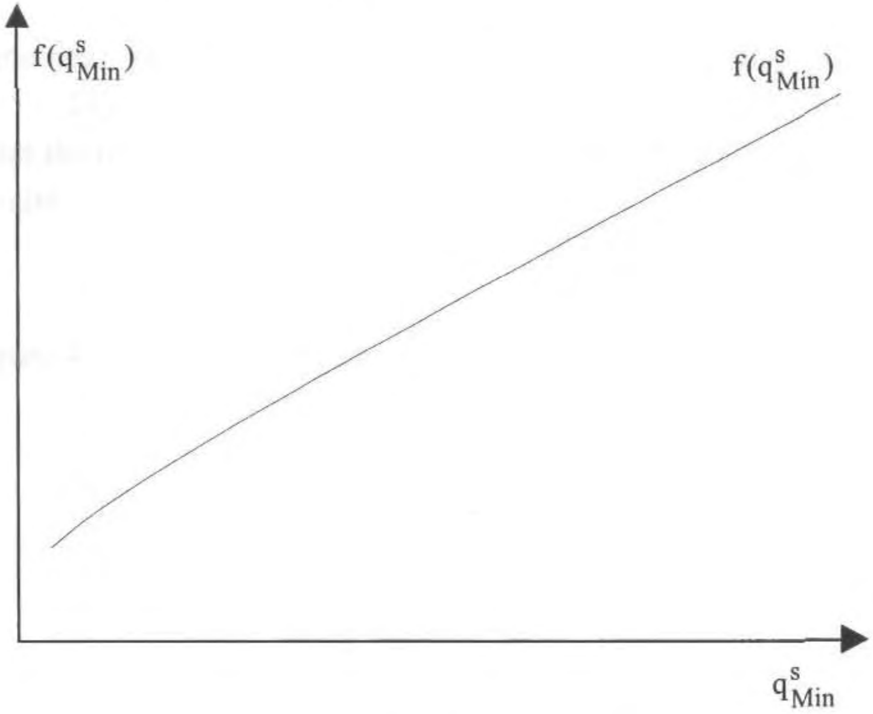

figure 3: Course of the function $f(q^S_{Min})$

In order to facilitate the understanding of our further elaboration we repeat here the function $f(Q, q^S_{Min})$.

$$f(Q, q^S_{Min}) = \frac{Q}{\ln\left[\frac{Q}{q^S_{Min}} \cdot (\delta_S - 1) + 1\right]} \quad (19)$$

Furthermore, before we proceed to generalize our analytical results it is important to note the following fact. Consider two given values of Q denoted by Q' and Q'' with $Q'' = 2 \cdot Q'$ as well as a given value q^S_{Min}. If we increase Q' to Q'' then the nominator in (19) is doubled. However, in the same situation the denominator increases by less than the twofold. Therefore, we can deduce that for any value of q^S_{Min} the statement given below is valid:

$$f(Q', q^S_{Min}) < f(Q'', q^S_{Min}) \qquad \forall q^S_{Min} \quad (39)$$

Like in the previous subsection the figure 4 shows the course of the function $f(Q, q^S_{Min})$ for three different given values of Q denoted by Q_1; Q_2 and Q_3 with $Q_1 < Q_2 < Q_3$.

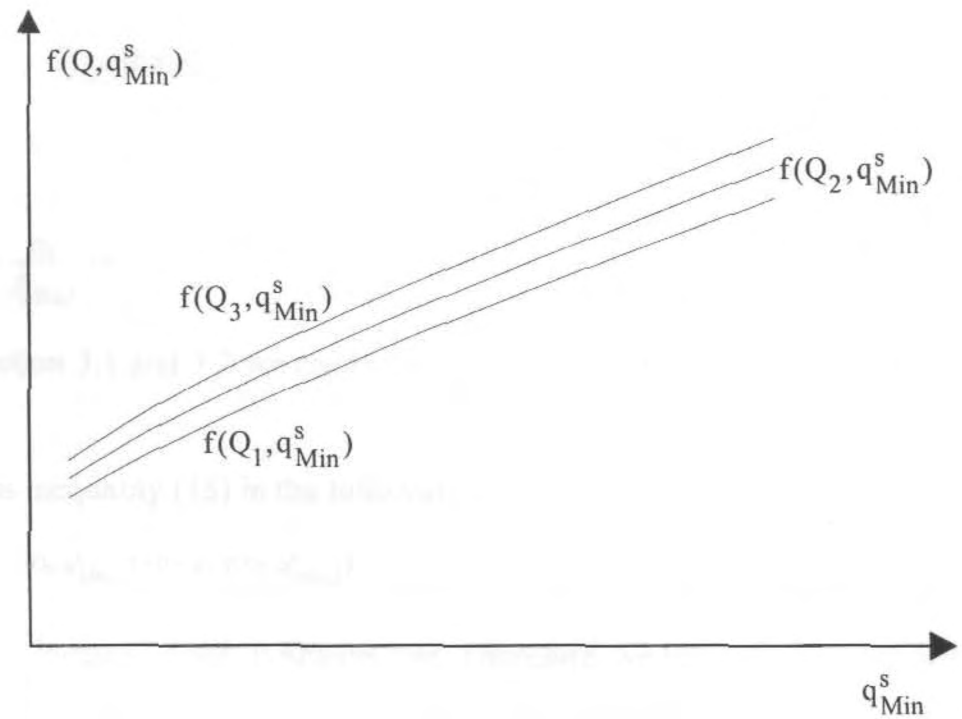

figure 4: Course of the function $f(Q, q^S_{Min})$ for three different values of Q

3.3 Generalization of the Analytical Results

In this subsection we use a grafical proof in order to verify the validity of inequality (15). Recall inequality (15) was given by:

$$\frac{[\lambda \cdot Q_1 + (1-\lambda)Q_2]}{\ln\left[\frac{\lambda \cdot Q_1 + (1-\lambda)Q_2}{\lambda \cdot q^S_{Min,1} + (1-\lambda)\cdot q^S_{Min,2}} \cdot (\delta_S - 1) + 1\right]} \geq \lambda \cdot \frac{Q_1}{\ln\left[\frac{Q_1}{q^S_{Min,1}} \cdot (\delta_S - 1) + 1\right]} + (1-\lambda) \cdot \frac{Q_2}{\ln\left[\frac{Q_2}{q^S_{Min,2}} \cdot (\delta_S - 1) + 1\right]} \qquad (15)$$

Using the results derived in subsection 3.1 and 3.2 we can depict the course of the function $f(Q, q^S_{Min})$ with respect to the two variables Q and q^S_{Min} like it figure 5 shows.

By using $f(Q, q^S_{Min})$ we can express inequality (15) in the following manner:

$$f(\lambda \cdot Q_1 + (1-\lambda) \cdot Q_2; \lambda \cdot q^S_{Min,1} + (1-\lambda) \cdot q^S_{Min,2}) \geq \lambda \cdot f(Q_1; q^S_{Min,1}) + (1-\lambda) \cdot f(Q_2; q^S_{Min,2}) \qquad (40)$$

Like it can be seen from figure 5 the inequality (40) is always true. Therefore, we have also proved the validity of inequality (15).

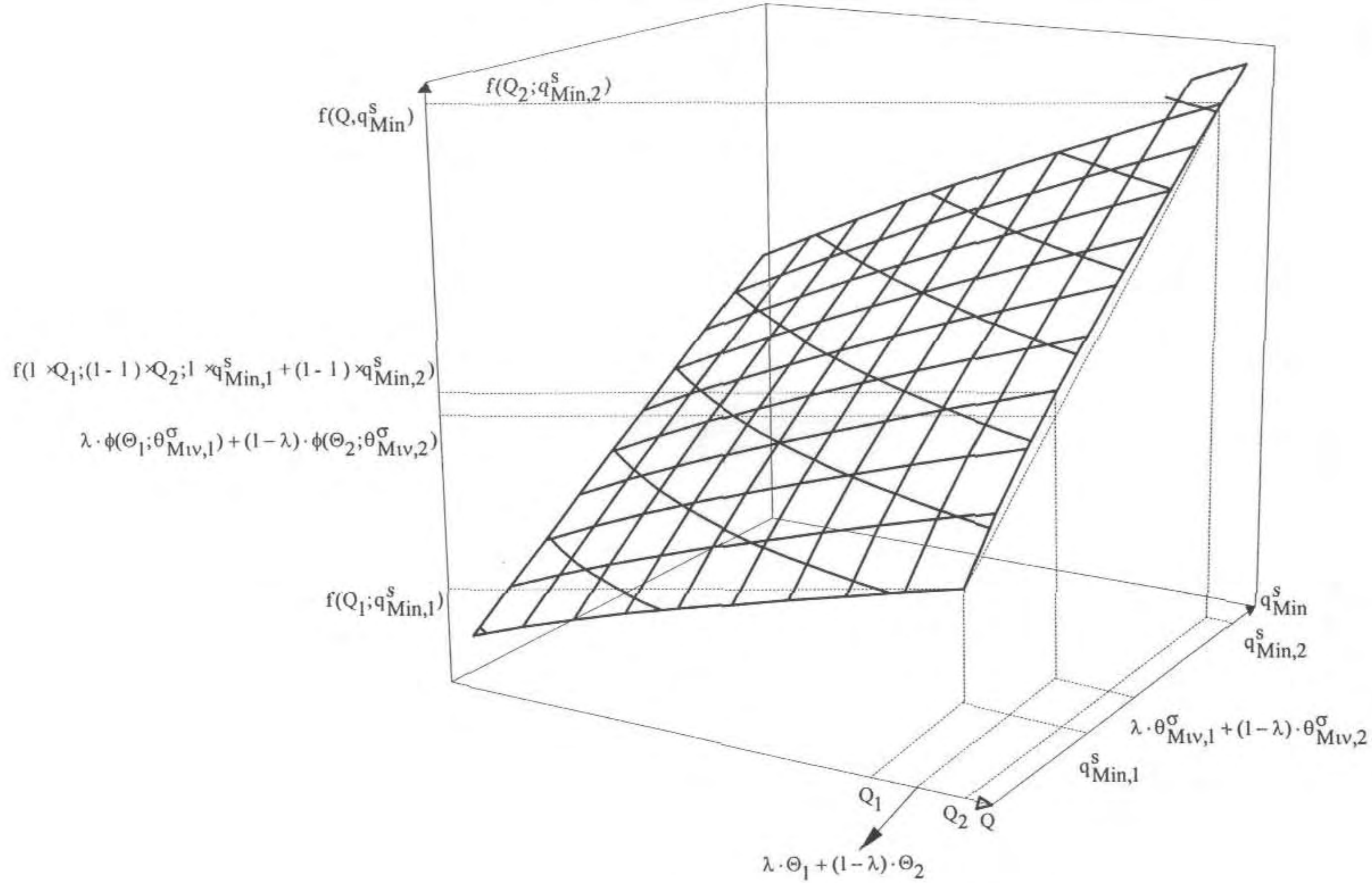

figure 5: Course of the function $f(Q, q^S_{Min})$

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