Uncertainty in the Black-Litterman Model: Empirical Estimation of the Equilibrium

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Abstract

The Black-Litterman model is a widely used and well established application of the Bayesian framework to asset allocation problems. It is, however, difficult to calibrate, as it requires the specification of abstract uncertainty parameters. We propose a new, more flexible model that allows the empirical estimation of the equilibrium, alleviating the need for parametrization. In an empirical application, we illustrate the sensitivity of the classical Black-Litterman model to the choice of the uncertainty parameter. We then demonstrate that the flexible model successfully exploits information in the cross-section of index constituents' returns to find an optimal trade-off in calibration of the uncertainty.

Contents

1 Introduction

Deriving an optimal asset allocation hinges crucially on the quality of inputs used in the optimization. If the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ are known with certainty, the classical mean-variance optimization of Markowitz (1952) produces optimal portfolios. If, however, both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are estimated with uncertainty, mean-variance optimization tends to maximize estimation error, as shown in Michaud (1989).

The Black-Litterman model (Black and Litterman (1990, 1991, 1992)), a derivation of the Bayesian methods developed in academia, has particular practical appeal for investors. It allows the specification of subjective views and an uncertainty about these views, which are combined with equilibrium returns and incorporated consistently to arrive at $\tilde{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\Sigma}}$. These new parameters can then be used in the mean-variance portfolio optimization process. In the Black-Litterman model, however, uncertainty about the equilibrium returns is specified with an overall scalar uncertainty parameter τ, which is abstract and difficult to set.

We propose a slight deviation from the Black-Litterman model that leads to two significant changes: First, the model allows for more flexibility, as the uncertainty about equilibrium returns is specified individually for each asset, not with a single scalar parameter. Second, it allows the empirical estimation of this flexible uncertainty from the data. We argue that the second feature is necessary for the model to comply with the theory of semi-strong-form market efficiency, which is violated in the classical Black-Litterman model, and develop the full theoretical data model implied by our deviation. This data model is based on the notion of random intercepts characterised by distributional parameters. The distributional parameters then carry information about the uncertainty in the equilibrium model.

Building on the theoretical foundations, we then illustrate how the relevant parameters can be estimated using a one-way stratified error components model, a method developed for panel data analysis that exploits within-group variations of estimated intercepts to characterise their distribution. Like that, we can decompose the uncertainty about the equilibrium returns into two components, uncertainty about the equilibrium model and estimation error. We show that common specifications of the classical Black-Litterman model only take estimation error into account.

In a regional allocation problem for the European equities market, we illustrate the application of the flexible model to data and compare it to different specifications of the classical Black-Litterman model. We find that, if no subjective views are specified, the flexible model achieves the best performance with an annualized performance of 0.15% relative to the capitalization weighted benchmark. Black-Litterman specifications achieve negative relative returns under the same conditions. Indeed, we find that the flexible model achieves a statistically significant Jensen's Alpha of 0.45% per year when compared to the most common specification of the classical Black-Litterman model.

To investigate the performance of the flexible model in the presence of subjective views, we conduct a simulation exercise in the spirit of Gofman and Manela (2012), but introduce a new bias factor to simulate cases where the views are incorrect. The classical Black-Litterman models produce results sensitive to the choice of the uncertainty parameter chosen by the investor. We can show that the flexible model, on the other hand, achieves balanced results under most conditions, indicating a good trade-off in the choice of equilibrium uncertainty.

2 Literature Review

While the seminal work by Markowitz (1952) laid the foundations for modern portfolio theory, implementation by practitioners has not been widespread (Michaud (1989) dubbed this the "Markowitz optimization enigma"). Resulting portfolios are often highly-concentrated, very sensitive to the input parameters and maximize estimation error in the inputs (Idzorek (2007)). These related problems predominately stem from the assumption that the mean vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ used for mean-variance optimization are stable and known with certainty. In fact, these input parameters are unknown and can only be estimated with uncertainty, which has to be taken into account in portfolio optimization. Jobson and Korkie (1980, 1981) document this problem. They show that simple equal-weighting actually outperforms mean-variance optimization in the presence of estimation uncertainty in input parameters. Michaud (1989) states that mean variance optimization actually maximizes estimation risk. He highlights several additional practical problems that are associated with this fact. Also highlighting practical issues with mean-variance optimization, Best and Grauer (1991) show how sensitive portfolio weights react to the estimated mean returns.

To deal with parameter uncertainty, the Bayesian framework has gained popularity. It allows to optimally combine two sets of information, usually sample and non-sample data (Rachev et al. (2008)). A prior belief is updated with new data (from the sample or other sources of information) and optimally combined to the posterior distribution. As an intuition, Bayesian methods take into account that a view on one parameter of the model affects all other parameters as well.[1] For the problem of portfolio selection, Bayesian methods are used to derive updated posterior distributions of returns.

A particular application of the Bayesian framework in this context is the Black-Litterman model, proposed by Black and Litterman (1990, 1991, 1992) and further discussed in He and Litterman (1999). According to Rachev et al. (2008), the Black-Litterman model is "the single most prominent application of the Bayesian methodology to portfolio selection". It allows to

[1] For a through discussion of the Bayesian framework, consider Rachev et al. (2008) or Scherer (2010).

consistently combine two sets of information: A market-equilibrium and the investors subjective views. This explains its practical appeal, as an investor does not need to rely entirely on either a quantitative, backward-looking model, nor a fully views-based framework, but is able to combine the two in a consistent way. To achieve that, the Black-Litterman model starts from an equilibrium asset pricing model (the Sharpe (1964) and Lintner (1965) Capital Asset Pricing Model CAPM), that is true only with a certain confidence. The equilibrium model is then combined with the investor's subjective views on asset returns using Bayesian methods.

Later extensions of the model explore different directions: To apply it to non-normal returns distributions, Giacometti et al. (2007) allow its application to the family of stable distributions. Going further, Meucci (2008) develops a procedure to apply it to fully general non-normal markets, which does, however, rely on simulation, as no analytical solution is available. On the other hand, Cheung (2010) and Kolm and Ritter (2020) develop methods that incorporate multi-factor models and allow the specification of views on these factors as well. A recent overview of extensions to the original Black-Litterman model, including robust optimization and multi-period extensions is provided in Kolm et al. (2021).

While these extensions offer the ability to apply the Black-Litterman model in a very general setting, a key question remains: How to specify the uncertainty in estimated equilibrium returns. In this regard, Satchell and Scowcroft (2007) propose a model where τ is stochastic. Also, Allaj (2013) proposes an econometric model to estimate the same parameter from the data. There is, however, to the best knowledge of the authors, no publication that derives an empirical model to estimate the uncertainty of estimated equilibrium returns individually for each asset.

3 Theory

This theoretical part is concerned with outlining the basic theoretical foundations and the deviations that we propose. Besides reproducing the well-known Black-Litterman model (Section 3.1) and discussing the uncertainty parameters involved (Section 3.2), we will propose a more flexible version of the model in Section 3.3 and relate it to the theoretical notion of market efficiency (Section 3.4). The flexible model will be studied and compared to the classical Black-Litterman model by investigating the properties of the implied data models in Section 3.5.

3.1 The Black-Litterman model

In the Black-Litterman model (Black and Litterman, 1990, 1991, 1992), the assets' excess returns $\boldsymbol{r}$ (a $(N \times 1)$ vector of random variables, where N is the number of assets) are assumed to come from a multivariate normal distribution:

$$\boldsymbol{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{1}$$

where $\boldsymbol{\mu}$ is the $(N\times 1)$ vector of expected excess returns, and $\boldsymbol{\Sigma}$ is the $(N\times N)$ covariance-matrix. The model then exploits two additional sources of information: First, the parameter $\boldsymbol{\mu}$ is expected to contain estimation error, as it is unknown. Second, subjective views on the assets expected excess returns are specified in the $(K\times 1)$ vector $\boldsymbol{q}$, where K is the number of views. The Black-Litterman model provides adjusted parameters that incorporate this information, such that the posterior distribution is

$$\boldsymbol{r}|\boldsymbol{q} \sim N(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}) \tag{2}$$

In what follows, the derivation of $\tilde{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\Sigma}}$ in the Black-Litterman model is discussed. These updated parameters then serve as the inputs for traditional mean-variance optimization, to derive optimal allocation weights.

Equilibrium Returns In the classical Black-Litterman model, the $(N\times 1)$ vector of equilibrium returns $\boldsymbol{\pi}$ is derived either through reverse optimization (Sharpe, 1974) or from the Sharpe (1964) and Lintner (1965) CAPM:

$$\boldsymbol{\pi} = \delta \boldsymbol{\Sigma} \boldsymbol{\omega}_m = \boldsymbol{\beta} \mu_m, \tag{3}$$

where δ is the scalar risk-aversion coefficient, $\boldsymbol{\omega}_m$ is the $(N\times 1)$ vector of market capitalization weights, $\boldsymbol{\beta}$ is the $(N\times 1)$ vector of market betas of the assets and μ_m is the scalar market risk premium.[2] Although the market is expected to be in equilibrium on average, at any given point in time it could be in disequilibrium. Therefore,

$$\boldsymbol{\mu} = \boldsymbol{\pi} + \boldsymbol{\epsilon} \tag{4}$$

$$\text{with} \qquad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \boldsymbol{\Psi})$$

where $\boldsymbol{\epsilon}$ is a $(N\times 1)$ vector of random shocks that push the market off its long-run equilibrium and $\boldsymbol{\Psi}$ is the $(N\times N)$ covariance matrix of these shocks.

[2] Rachev et al. (2008) show that the two approaches are equivalent under the assumption that capital markets are in equilibrium and clear. Consider also Idzorek (2007) and Satchell and Scowcroft (2007) for additional details.

Combining these expressions yields the prior distribution of $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} \sim N(\boldsymbol{\pi}, \boldsymbol{\Psi}) \tag{5}$$

The prior covariance matrix $\boldsymbol{\Psi}$ represents the uncertainty in the accuracy with which $\boldsymbol{\pi}$ is estimated. It is an important hyperparameter and the main focus of our paper.

Investor Views One of the main advantages of the Black-Litterman model is the possibility for the investor to specify subjective views on the absolute or relative performance of assets. These views are specified using the views matrix $\boldsymbol{P}$, a $(K \times N)$ matrix of K views on N assets; a vector $\boldsymbol{q}$ of expected excess returns of the K views; and a $(K \times K)$ covariance matrix $\boldsymbol{\Omega}$ of these views. $\boldsymbol{P}$ and $\boldsymbol{q}$ are of the following form:

$$\boldsymbol{P} = \begin{bmatrix} p_{1,1} & \cdots & p_{1,N} \\ \vdots & \ddots & \vdots \\ p_{K,1} & \cdots & p_{K,N} \end{bmatrix} \qquad \boldsymbol{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_K \end{bmatrix}$$

Every line of $\boldsymbol{P}$ specifies a long-short (or long-only) portfolio of the N assets, with every $p_{k,i}$ specifying the weight of the ith asset $(i = 1, \ldots, N)$ in the kth view $(k = 1, \ldots, K)$. Every element of $\boldsymbol{q}$ then specifies the expected return of the respective portfolio. The returns of the views are also assumed to be uncertain, such that:

$$\boldsymbol{q} = \boldsymbol{P}\boldsymbol{\mu} + \boldsymbol{\varepsilon} \tag{6}$$

$$\text{where} \qquad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \boldsymbol{\Omega}),$$

with $\boldsymbol{\varepsilon}$ a $(K \times 1)$ vector of random shocks.

For the covariance matrix of the shocks, a simplifying assumption is usually made: The views are assumed to be uncorrelated, i.e. the covariance matrix $\boldsymbol{\Omega}$ is assumed to be diagonal (with zeros on all off-diagonal elements):

$$\boldsymbol{\Omega} = \begin{bmatrix} \omega_{1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_{K,K} \end{bmatrix} \tag{7}$$

The elements $\omega_{k,k}$ encompass the uncertainty about the views. They should be inversely proportional to the strength of the investor's confidence in the kth view. Different ways to determine $\omega_{k,k}$ will be discussed in Section 3.2.

With $\boldsymbol{P}$, $\boldsymbol{q}$ and $\boldsymbol{\Omega}$ thus defined, the prior distribution of the view's expected returns is assumed to be a multivariate normal distribution of the form:

$$\boldsymbol{q}|\boldsymbol{\mu} \sim N(\boldsymbol{P\mu}, \boldsymbol{\Omega}) \tag{8}$$

Combining the prior distributions Using Bayes' theorem, the two sources of information can be combined consistently. According to Bayes' theorem, the following relationship holds:

$$f(\boldsymbol{\mu}|\boldsymbol{q}) \propto f(\boldsymbol{q}|\boldsymbol{\mu})\, f(\boldsymbol{\mu}) \tag{9}$$

Then, the posterior distribution of expected excess returns $\boldsymbol{\mu}$ given subjective views $\boldsymbol{q}$ is[3]

$$\boldsymbol{\mu}|\boldsymbol{q} \sim N(\boldsymbol{m}, \boldsymbol{V}), \tag{10}$$

$$\text{where} \quad \boldsymbol{m} = \boldsymbol{V}\left(\boldsymbol{\Psi}^{-1}\boldsymbol{\pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{q}\right) \tag{11}$$

$$\text{and} \quad \boldsymbol{V} = \left(\boldsymbol{\Psi}^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right)^{-1} \tag{12}$$

The posterior distribution of the assets' returns given the investor's views is then given by:

$$f(\boldsymbol{r}|\boldsymbol{q}) = \int f(\boldsymbol{r}|\boldsymbol{\mu})\, f(\boldsymbol{\mu}|\boldsymbol{q})\, d\boldsymbol{\mu} \tag{13}$$

This yields a multivariate normal distribution of the form outlined in Equation (2):

$$\boldsymbol{r}|\boldsymbol{q} \sim N(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}) \tag{14}$$

$$\text{with} \quad \tilde{\boldsymbol{\mu}} = \boldsymbol{m} \qquad \text{and} \qquad \tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} + \boldsymbol{V}$$

Note that the new covariance matrix $\tilde{\boldsymbol{\Sigma}}$ is simply the sample covariance matrix $\boldsymbol{\Sigma}$ increased by the uncertainty surrounding the estimate of $\tilde{\boldsymbol{\mu}}$, captured by $\boldsymbol{V}$.

3.2 Hyperparameters

In the Black-Litterman model, two hyperparameters are used to specify the investor's confidence in the equilibrium returns on the one hand and his own views on the other. They are captured by $\boldsymbol{\Psi}$ and $\boldsymbol{\Omega}$, respectively. These parameters express the uncertainty about the expected excess equilibrium returns $\boldsymbol{\pi}$ and the expected excess returns of the view-portfolios $\boldsymbol{q}$ that enter the model. As shown in Scherer (2010), these parameters control the mixing of the two sources of information, where more importance is attributed to the source with less uncertainty.

[3] For a mathematical proof, consider for instance Satchell and Scowcroft (2007). Also, Kolm and Ritter (2017) provide a complete Bayesian derivation of the model.

As pointed out by Idzorek (2007), these parameters are “the most abstract and difficult to specify parameters of the model”. As a consequence, various methods to determine their values have emerged from the literature. The majority of these methods focus on setting $\mathbf{\Omega}$, the uncertainty about the expected excess returns of the investor’s subjective views.

Setting Hyperparameter $\mathbf{\Psi}$ In Black and Litterman (1991, 1992), the original authors reduce the problem of setting $\mathbf{\Psi}$ to setting a single scalar parameter τ, as they assume that $\mathbf{\Psi}$ is proportional to the covariance matrix $\mathbf{\Sigma}$:

$$\mathbf{\Psi} = \tau\mathbf{\Sigma} \tag{15}$$

The authors recommend to use a τ that is smaller than one and close to zero, as the mean of expected returns can much more accurately be determined than the expected returns themselves.
Subsequent literature focuses on how to set τ, that is, the assumption of proportionality to the empirical covariance matrix is not relaxed. Idzorek (2007) interprets τ as the inverse of the relative weight given to the equilibrium weights, or alternatively, inverse to the degree of belief in the equilibrium model. He also reports that practitioners recommend using a value of τ between 0.01 and 0.05. On the other hand, Satchell and Scowcroft (2007) propose to use a value of 1 for τ. Finally, Rachev et al. (2008) and Meucci (2010) suggest to use the standard error of the estimate of the implied equilibrium returns directly, which is approximately 1 divided by the number of observations. They do, however, also state that “no guideline exists for the selection of their values”.
Although suggestions on how to set τ are available, there is still a lot of subjectivity. It would be beneficial to either make this subjectivity more explicit or to derive empirical methods to determine τ. Allaj (2013) makes a contribution in this direction by developing an empirical model to estimate the value of τ from the data. Additionally, he investigates the proportionality assumption of Black and Litterman (1991, 1992) and suggests that $\mathbf{\Psi}$ should not be proportional to the empirical covariance matrix $\mathbf{\Sigma}$, but rather to the idiosyncratic risk which is the covariance matrix conditional on the equilibrium model, an observation also stated in Gofman and Manela (2012).

Setting Hyperparameter $\mathbf{\Omega}$ Also for setting $\mathbf{\Omega}$, Black and Litterman (1991, 1992) make a simplifying assumption: As views are assumed to be independent of each other, $\mathbf{\Omega}$ reduces to a diagonal matrix (as shown in Equation (7)). They propose to express the uncertainty or, conversely, the confidence in a view as the hypothetical number of observations drawn from the distribution of future returns. Alternatively, if the view is assumed to directly specify a

probability distribution, a variance or volatility of the view can be specified.
Several alternative procedures have been proposed to estimate $\mathbf{\Omega}$:

- In order to determine each element $\omega_{k,k}$ of $\mathbf{\Omega}$, He and Litterman (1999) propose to use the same τ as used in the estimation of $\mathbf{\Psi}$ (see above). The variance of the kth view portfolio is simply $\boldsymbol{p}_k \mathbf{\Sigma} \boldsymbol{p}'_k$.[4] It is then scaled with the same constant τ, thus resulting in:

$$\omega_{k,k} = \tau \boldsymbol{p}_k \mathbf{\Sigma} \boldsymbol{p}'_k \tag{16}$$

  Peterson (2012) uses the same approach, but with $\tau = 1$.

- If views are derived from a quantitative forecasting model, Scherer (2010) proposes to use the unexplained variances of the respective forecasting model for each $\omega_{k,k}$:

$$\omega_{k,k} = \sigma_k^2 \left(1 - R_k^2\right), \tag{17}$$

  where R_k^2 is the coefficient of determination of the forecasting model. Each $\omega_{k,k}$ is then the view's variance conditioned on the forecasting model.

- Satchell and Scowcroft (2007) use Bayesian methods to allow for a prior belief on the covariance matrix as well. Similarly, Idzorek (2007) proposes an iterative process that uses certainty-equivalent weights and chooses the portfolio weights such that they represent a confidence specified by the investor.

- Rachev et al. (2008) and Scherer (2010) propose a purely judgemental approach: Implied variances of views are derived from the investors judgement using a confidence interval approach. As the Black-Litterman model assumes the returns of the views-portfolios to be independently normally distributed, the investor can be asked to characterize the normal distribution of each view by specifying an expected interval and a confidence that returns will lie in this interval.

The review shows that considerably more work has been published on how to set the uncertainty in the expected returns of the investor's subjective views than on the the uncertainty of the expected equilibrium returns. The aim of this paper is to contribute to filling this gap: First, we introduce more flexibility in the specification of uncertainty in expected equilibrium returns by relaxing the proportionality assumption in Equation (15). Second, we propose an empirical estimation method for the flexible uncertainty parameters.

[4] $\boldsymbol{p}_k$ is the kth row of the views matrix $\boldsymbol{P}$

3.3 Deviation from the Black-Litterman model

As discussed in Section 3.2, setting the hyperparameters of the Black-Litterman model poses a challenge for practical applications. While several approaches are available to flexibly specify the uncertainty about the expected returns of the views $\mathbf{\Omega}$, uncertainty about the expected equilibrium returns $\mathbf{\Psi}$ is specified through a single parameter τ that forces proportionality to the covariance matrix $\mathbf{\Sigma}$. We propose a simple deviation of the Black-Litterman model to allow a more flexible specification of $\mathbf{\Psi}$.

Introducing Flexibility The classical Black-Litterman model assumes that the uncertainty about equilibrium returns is proportional to the uncertainty about the returns themselves (Black and Litterman (1991, 1992)), as formalized in Equation (15).

We would like to allow for more flexibility, still encompassing the Black-Litterman specification as a special case. In order to do this, first decompose the covariance matrix of the returns $\mathbf{\Sigma}$ into a $(N\times 1)$ volatility vector $\boldsymbol{\sigma}$ and a $(N\times N)$ correlation matrix $\mathbf{\Phi}$:

$$\mathbf{\Sigma} = \mathrm{diag}(\boldsymbol{\sigma})\mathbf{\Phi}\mathrm{diag}(\boldsymbol{\sigma}) \qquad \text{or} \qquad \mathbf{\Phi} = \mathrm{diag}(\boldsymbol{\sigma})^{-1}\mathbf{\Sigma}\mathrm{diag}(\boldsymbol{\sigma})^{-1} \tag{18}$$

Note that $\boldsymbol{\sigma}$ is simply the square root of the diagonal elements of $\mathbf{\Sigma}$. This allows us to retain the information about the co-movement of equilibrium returns, but leaves flexibility in their uncertainty. Define the $(N\times 1)$ vector of standard errors of estimated equilibrium returns chosen by the investor as $\boldsymbol{\sigma_\pi}$, then

$$\mathbf{\Psi} = \mathrm{diag}(\boldsymbol{\sigma_\pi})\mathbf{\Phi}\mathrm{diag}(\boldsymbol{\sigma_\pi}) \tag{19}$$

To see how this is a generalization of the Black-Litterman approach, suppose the investor sets $\boldsymbol{\sigma_\pi} = \sqrt{\tau}\boldsymbol{\sigma}$ (i.e. he sets the standard errors of estimated equilibrium returns to be proportional to the empirical volatilities of returns). Then, we recover

$$\mathbf{\Psi} = \mathrm{diag}(\boldsymbol{\sigma_\pi})\mathbf{\Phi}\mathrm{diag}(\boldsymbol{\sigma_\pi}) = \mathrm{diag}(\sqrt{\tau}\boldsymbol{\sigma})\mathbf{\Phi}\mathrm{diag}(\sqrt{\tau}\boldsymbol{\sigma}) = \tau\mathbf{\Sigma} \tag{20}$$

but do not restrict the investor to this case.
With this decomposition, the investor has the ability to specify the uncertainty about equilibrium returns for each asset individually.

Correspondence to τ As the parameter τ used in the classical Black-Litterman model is simply the parameter of proportionality between $\mathbf{\Psi}$ and $\mathbf{\Sigma}$, it can be computed from the specified $\boldsymbol{\sigma_\pi}$. For each asset i, given the empirical variance of the returns σ_i^2 and the specified variance of the estimate of the equilibrium return, $\sigma^2_{\pi,i}$, τ_i is simply:

$$\tau_i = \frac{\sigma^2_{\pi,i}}{\sigma^2_i} \tag{21}$$

Note that we index τ over i as well, as with the added flexibility, τ can be different for every asset.

3.4 Market Efficiency

To reconcile the Black-Litterman model with financial theory, consider the taxonomy of market efficiency proposed by Fama (1970). He identifies three forms of market efficiency:

- *Weak-form market efficiency* states that all available information contained in historical market returns is reflected in current market prices.
- *Semi-strong-form market efficiency* states that all information that is obviously publicly available is reflected in current market prices.
- *Strong-form market efficiency* states that all information (public and private) is reflected in current market prices.

In the case of strong-form market efficiency, it is not possible for an individual investor to achieve superior returns over the market portfolio by using private information, since all information is already reflected in market prices. In fact, this is the basic assumption of the Sharpe (1964) and Lintner (1965) CAPM, as outlined in Fama and French (2004): All investors have homogeneous information sets, and draw the same conclusions form them.
In contrast, the distinguishing feature of the Black-Litterman model is to relax this assumption. It assumes semi-strong-form market efficiency and specifically allows to combine all available public information (through the equilibrium model) with a private information set (the investor's subjective views), potentially leading to superior returns (Cheung, 2010; Allaj, 2013).

It follows that the distribution of expected equilibrium returns in the Black-Litterman model constitutes the public information set, and indeed the original authors (Black and Litterman, 1991, 1992) state about this distribution: "We assume this information is known to all; it is not a function of the circumstances of any individual investor." Now, recall that this distribution as defined in Equation (5) has the form:

$$\boldsymbol{\mu} \sim N(\boldsymbol{\pi}, \boldsymbol{\Psi}), \tag{22}$$

with two proposed specifications for $\boldsymbol{\Psi}$:

- The classical approach: $\boldsymbol{\Psi} = \tau\boldsymbol{\Sigma}$
- The flexible approach: $\boldsymbol{\Psi} = \text{diag}(\boldsymbol{\sigma_\pi})\boldsymbol{\Phi}\text{diag}(\boldsymbol{\sigma_\pi})$

But then, to conform with the assumption of semi-strong-form market efficiency, Expression (22) can contain only publicly available information. So, neither τ nor $\boldsymbol{\sigma_\pi}$ can be subjective parameters; they have to be public information as well.

For the classical approach, two of the propositions discussed in Section 3.2 actually comply with Black and Litterman's assumption of semi-strong-form market efficiency:

- Rachev et al. (2008) and Meucci (2010) propose to set $\tau = \frac{1}{T}$, where T is the number of observations used to estimate the equilibrium returns. As will be shown in Section 3.5, this is a sensible choice in terms of econometric modelling. While it complies with the assumption of semi-strong-form market efficiency, as T is clearly an observable parameter, it is still not entirely objective. The choice of the length of the dataset used can be subjective, as a trade-off between a higher number of observations and the relevance of the data might emerge. Also, note that when $T \to \infty$, τ tends to zero. Recall that $\boldsymbol{\Psi}$ measures the size of the shocks that can throw the market off it long run equilibrium. It is not clear why the use of more and more data to estimate the equilibrium model should reduce the size of these shocks.

- Allaj (2013) proposes an econometric model that allows the estimation of τ from the data. As all the parameters of the distribution of equilibrium returns are estimated empirically, they are subjective and this approach fully complies with the assumption of semi-strong-form market efficiency.

The question then arises whether it is possible to estimate the proposed flexible model from the data, yielding a procedure that is consistent with the assumption of semi-strong-form market efficiency. This will be the topic of the following sections.

3.5 Implied Data Models

In this section the econometric models of the actual returns data are studied. The main objective is to state the required assumptions and how the data models tie back to the expected equilibrium returns of the Black-Litterman model. This is done twice, first for the classical model, relying on the CAPM, and a second time for the data model permitted by the more flexible specification presented prior. It is important to note that, as these are models of the data only and do not require additional subjective parameters, they comply with the assumption of semi-strong-form market efficiency. How to estimate these econometric models will be the subject of Section 4.

The Classical Model In the classical framework, the data model is based on work by Fama and MacBeth (1973). It relies on the following underlying panel data model:

$$r_{it} = \alpha_i + \beta_i r_{mt} + \varepsilon_{it} \qquad \text{with} \quad \varepsilon_{it} \sim N(0, \sigma^2_{\varepsilon_i}), \tag{23}$$

for $t = 1, \dots, T$, $i = 1, \dots, N$. r_{it} is the excess return of asset i at time t, r_{mt} is the excess return of the market portfolio at time t (where excess returns are returns above the risk-free interest rate), α_i is an unknown constant pertaining to asset i, β_i is the unknown CAPM-beta of asset i, and ε_{it} is a random disturbance.
The following assumptions on the structure of the error terms are required:

$$\begin{aligned} \mathbb{E}[\varepsilon_{it}] &= 0 \qquad \mathbb{E}\left[\varepsilon^2_{it}\right] = \sigma^2_{\varepsilon_i} \qquad \mathbb{E}[\varepsilon_{it}\varepsilon_{jt}] = \sigma_{\varepsilon_{ij}} \\ \mathbb{E}[\varepsilon_{it}\varepsilon_{is}] &= 0 \qquad \mathbb{E}[\varepsilon_{it}\varepsilon_{js}] = 0 \end{aligned} \tag{24}$$

Thus, the error terms are assumed to be contemporaneously correlated, but serially independent. Additionally, the market returns are assumed to be temporally i.i.d., such that $\mathbb{E}[r_{mt}r_{ms}] = \mathbb{E}[r_{mt}]\mathbb{E}[r_{ms}]$. The data model then implies the following properties for the excess returns:

$$\begin{aligned} \operatorname{Var}[r_{it}] &= \beta_i^2\sigma_m^2 + \sigma^2_{\varepsilon_i} = \sigma_i^2 & \operatorname{Var}[r_{it} \mid \boldsymbol{r_m}] &= \sigma^2_{\varepsilon_i} \\ \operatorname{Cov}[r_{it}, r_{jt}] &= \beta_i\beta_j\sigma_m^2 + \sigma_{\varepsilon_{ij}} = \sigma_{ij} & \operatorname{Cov}[r_{it}, r_{jt} \mid \boldsymbol{r_m}] &= \sigma_{\varepsilon_{ij}} \\ \operatorname{Cov}[r_{it}, r_{is}] &= 0 & \operatorname{Cov}[r_{it}, r_{is} \mid \boldsymbol{r_m}] &= 0 \\ \operatorname{Cov}[r_{it}, r_{js}] &= 0 & \operatorname{Cov}[r_{it}, r_{js} \mid \boldsymbol{r_m}] &= 0, \end{aligned}$$

where σ_m^2 is the variance of the excess returns of the market, σ_i^2 is the variance of the excess returns of asset i and σ_{ij} is the covariance of the excess returns of assets i and j.

Relation to Black-Litterman (Classical Model) The Black-Litterman model states a relationship between the expected excess returns $\boldsymbol{\mu}$ and the expected equilibrium returns $\boldsymbol{\pi}$ as in Equation (4). Combining this with Equation (3) and relying on the CAPM yields the following basic relation:

$$\boldsymbol{\mu} = \boldsymbol{\beta}\mu_m + \boldsymbol{\epsilon} \tag{25}$$
$$\text{with} \qquad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \boldsymbol{\Psi})$$

which can be combined (as in Equation (5)) to:

$$\boldsymbol{\mu} \sim N\left(\boldsymbol{\beta}\mu_m, \boldsymbol{\Psi}\right) \tag{26}$$

To relate this expression to the data model just presented, recall that the best estimator for the expected excess return is the sample average of the observed excess returns, i.e. each element μ_i in the vector $\boldsymbol{\mu}$ is best estimated by the sample average $\bar{r}_i = \frac{1}{T}\sum_{t=1}^{T} r_{it}$. It is thus possible to relate the Black-Litterman model to the data model by investigating the implied properties of the sample averages $\bar{r}_i$.

The sample average of the excess returns for each asset i according to the data model is:

$$\bar{r}_i = \alpha_i + \beta_i \bar{r}_m + \bar{\varepsilon}_i \tag{27}$$

This shows that here, a key assumption has to be made: In the classical approach, the CAPM is assumed to hold for all assets, i.e. $\alpha_i = 0$, $\forall i$. Under this assumption, the sample average of the excess returns becomes:

$$\bar{r}_i = \beta_i \bar{r}_m + \bar{\varepsilon}_i \tag{28}$$

and the relation to the Black-Litterman model is established. It has the following properties:

$$\begin{aligned} \mathbb{E}[\bar{r}_i] &= \beta_i \mu_m \\ \mathrm{Var}[\bar{r}_i] &= \tfrac{1}{T}\left(\beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2\right) = \tfrac{1}{T}\sigma_i^2 \\ \mathrm{Cov}[\bar{r}_i, \bar{r}_j] &= \tfrac{1}{T}\left(\beta_i \beta_j \sigma_m^2 + \sigma_{\varepsilon_{ij}}\right) = \tfrac{1}{T}\sigma_{ij} \end{aligned} \tag{29}$$

In matrix terms, this can be expressed as:

$$\bar{\boldsymbol{r}} \sim N\left(\boldsymbol{\beta}\mu_m, \tfrac{1}{T}\boldsymbol{\Sigma}\right) \tag{30}$$

and can directly be compared to Equation (26) above, as it is the data model equivalent of the same expression. It lends direct justification for the proportionality assumption of Black and Litterman (1991, 1992) and the specification of $\tau = \frac{1}{T}$, as proposed for instance by Rachev et al. (2008) and Meucci (2010).

Note, however, that in this specification the CAPM is assumed to hold as all $\alpha_i = 0$. Consequently, the expression $\frac{1}{T}\boldsymbol{\Sigma}$ contains only uncertainty about the estimation, not uncertainty about whether the CAPM holds. As $T \to \infty$, that is, as more and more data is used to estimate the model, the expression tends to zero.[5]

[5] The data model also reconciles with the argument made by Allaj (2013) and Gofman and Manela (2012) briefly discussed in Section 3.2. Equation (25) shows that in the Black-Litterman model, uncertainty about expected equilibrium returns arises from the idiosyncratic shocks $\boldsymbol{\epsilon}$. Thus, the following conditional model is justified:

$$\begin{aligned} \mathrm{Var}[\bar{r}_i \mid \boldsymbol{r_m}] &= \mathrm{Var}[\bar{\varepsilon}_i] = \tfrac{1}{T}\sigma_{\varepsilon_i}^2 \\ \mathrm{Cov}[\bar{r}_i, \bar{r}_j \mid \boldsymbol{r_m}] &= \mathrm{Cov}[\bar{\varepsilon}_i, \bar{\varepsilon}_j] = \tfrac{1}{T}\sigma_{\varepsilon ij} \end{aligned}$$

The Flexible Model In order to accommodate the more flexible approach, and to be able to capture the uncertainty about the equilibrium model, we propose a slight variation in the data model (similar to Allaj (2013)).
According to Jensen (1968), the constant α_i (Jensen's Alpha) in the data model of the previous section measures the departure of an asset's excess returns from the returns predicted by the equilibrium model. Thus, the α_i contain information about how well the equilibrium model explains the observed returns. In order to exploit this fact, we suggest the following random intercept model:

$$r_{it} = \beta_i r_{mt} + \alpha_i + \varepsilon_{it} \tag{31}$$
$$\text{with} \quad \alpha_i \sim N(0, \sigma^2_{\alpha_i}) \quad \text{and} \quad \varepsilon_{it} \sim N(0, \sigma^2_{\varepsilon_i}),$$

Note the difference to the data model specification in the preceding section: α_i was an unknown constant and subsequently set to zero, imposing the CAPM to hold. Now, it is assumed that α_i is a random variable. This is in accordance with the Black-Litterman model, which expects "[...] the market to be on average in equilibrium, [but] at any given point in time this equilibrium could be perturbed by shocks." (Rachev et al. (2008), p. 144). Therefore, the random intercepts are on average zero ($\mathbb{E}[\alpha_i] = 0$), but have a variance ($\text{Var}[\alpha_i] = \sigma^2_{\alpha_i}$), driving the market off its equilibrium. The variance $\sigma^2_{\alpha_i}$ measures exactly the uncertainty about the equilibrium model for each asset i.

The following assumptions are required in addition to the assumptions in (24) for consistency:

$$\mathbb{E}[\alpha_i \alpha_j] = \sigma_{\alpha_{ij}}, \qquad \mathbb{E}[\alpha_i \varepsilon_{it}] = 0, \qquad \mathbb{E}[\alpha_i \varepsilon_{jt}] = 0 \tag{32}$$

The data model then implies the following properties of the excess returns:

$$\begin{aligned}
\text{Var}[r_{it}] &= \beta_i^2 \sigma_M^2 + \sigma^2_{\varepsilon_i} + \sigma^2_{\alpha_i} = \sigma_i^2 + \sigma^2_{\alpha_i} & \text{Var}[r_{it} \mid \boldsymbol{r_m}] &= \sigma^2_{\varepsilon_i} + \sigma^2_{\alpha_i} \\
\text{Cov}[r_{it}, r_{jt}] &= \beta_i \beta_j \sigma_M^2 + \sigma_{\varepsilon_{ij}} + \sigma_{\alpha_{ij}} = \sigma_{ij} + \sigma_{\alpha_{ij}} & \text{Cov}[r_{it}, r_{jt} \mid \boldsymbol{r_m}] &= \sigma_{\varepsilon_{ij}} + \sigma_{\alpha_{ij}} \\
\text{Cov}[r_{it}, r_{is}] &= \sigma^2_{\alpha_i} & \text{Cov}[r_{it}, r_{is} \mid \boldsymbol{r_m}] &= \sigma^2_{\alpha_i} \\
\text{Cov}[r_{it}, r_{js}] &= \sigma_{\alpha_{ij}} & \text{Cov}[r_{it}, r_{js} \mid \boldsymbol{r_m}] &= \sigma_{\alpha_{ij}},
\end{aligned} \tag{33}$$

In contrast to the classical model, serial and cross-serial correlation of assets excess returns are explicitly accounted for and captured in the covariance structure of the stochastic intercepts. This is in line with tests for the validity of the CAPM based on the presence of serial correlation in asset returns as documented in Fama (1970).

In matrix terms, this yields the setting discussed in Allaj (2013) and Gofman and Manela (2012):

$$\bar{\boldsymbol{r}}|\boldsymbol{r_m} \sim N\left(\boldsymbol{\beta}\mu_m, \tfrac{1}{T}\boldsymbol{S}\right),$$

where $\boldsymbol{S}$ is the $(N \times N)$ covariance matrix of the error terms.

Relation to Black-Litterman (Flexible Model) In the same way as for the classical approach, it is possible to relate the data model to the Black-Litterman model by investigating the implied properties of the sample averages of excess returns $\bar{r}_i$. According to the random intercept data model, these are:

$$\bar{r}_i = \beta_i \bar{r}_m + \alpha_i + \bar{\varepsilon}_i \tag{34}$$

Note that there is no need to impose the model to hold by assuming $\alpha_i = 0$: As the intercepts are random with $\mathbb{E}[\alpha_i] = 0$, the model is only assumed to hold on average.
The sample averages of excess returns have the following properties according to the random intercept data model:

$$\begin{aligned} \mathbb{E}[\bar{r}_i] &= \beta_i \mu_m \\ \mathrm{Var}[\bar{r}_i] &= \sigma^2_{\alpha_i} + \tfrac{1}{T}\sigma^2_i \\ \mathrm{Cov}[\bar{r}_i, \bar{r}_j] &= \sigma_{\alpha_{ij}} + \tfrac{1}{T}\sigma_{ij} \end{aligned} \tag{35}$$

In matrix terms, this can be expressed as:

$$\bar{\boldsymbol{r}} \sim N\left(\boldsymbol{\beta}\mu_m, \boldsymbol{A} + \tfrac{1}{T}\boldsymbol{\Sigma}\right), \tag{36}$$

where $\boldsymbol{A}$ denotes the $(N \times N)$ covariance matrix of the random intercepts. As before, Equation (36) can directly be compared to Equation (26). It then follows that assuming the random intercept data model, $\boldsymbol{\Psi}$ corresponds to $\boldsymbol{A} + \frac{1}{T}\boldsymbol{\Sigma}$. The uncertainty in expected equilibrium returns is thus composed of two components: The uncertainty about the equilibrium model, as captured in $\boldsymbol{A}$, plus the uncertainty about the estimation. The latter component is the familiar expression found in the classical approach, and vanishes as $T \to \infty$. This is an intuitive result, as using more and more data to estimate the model would decrease the estimation error, but uncertainty about the equilibrium model is still present.
While these theoretical results imply a correlation structure different to the correlation structure of the empirical covariance matrix $\boldsymbol{\Sigma}$, estimation of the off-diagonal elements in $\boldsymbol{A}$ poses methodological challenges. For that reason, a more practical approach is to use the specification of $\boldsymbol{\Psi}$ outlined in Section 3.3. More specifically the above results can be used to specify each element of $\boldsymbol{\sigma_\pi}$ as:

$$\sigma_{\pi,i} = \sqrt{\sigma^2_{\alpha_i} + \tfrac{1}{T}\sigma^2_i} \tag{37}$$

Then, the parameter $\boldsymbol{\Psi}$ is given as in Equation (19):

$$\boldsymbol{\Psi} = \mathrm{diag}(\boldsymbol{\sigma_\pi})\boldsymbol{\Phi}\mathrm{diag}(\boldsymbol{\sigma_\pi})$$

To summarize, this section presented the main theoretical properties of the Black-Litterman model and tied them to models of the actual returns data. The relevant parameters of the model are highlighted and the next section will turn to the methodological procedures that allow estimating them from the data.

4 Methodology

The problem addressed in this section is to estimate the prior distribution of expected equilibrium returns empirically, without relying on subjective parameters and thus complying with the assumption of semi-strong-form market efficiency as made in the original works by Black and Litterman (1990, 1991, 1992). This can be formalized as characterising the distribution in Equation (5):

$$\boldsymbol{\mu} \sim N(\boldsymbol{\pi}, \boldsymbol{\Psi})$$

4.1 Estimation of the Classical Model

In Section 3.5 the following parameterisation of the prior distribution was derived from the data model for the classical approach (see Equation (30)):

$$\bar{\boldsymbol{r}} \sim N\left(\boldsymbol{\beta}\mu_m, \tfrac{1}{T}\boldsymbol{\Sigma}\right)$$

The objective is thus to find the empirical estimates $\hat{\boldsymbol{\beta}}$, $\hat{\mu}_m$ and $\hat{\boldsymbol{\Sigma}}$ of the theoretical parameters characterising the distribution.

To this end, a well-known two-stage procedure as outlined by Fama and MacBeth (1973) can be employed. In the first stage, the data model (23) can directly be estimated:

$$r_{it} = \alpha_i + \beta_i r_{mt} + \varepsilon_{it} \qquad \text{with} \quad \varepsilon_{it} \sim N(0, \sigma^2_{\varepsilon_i}), \tag{38}$$

Even though the structure of the error term is clearly non-spherical, the parameter estimates $\hat{\boldsymbol{\beta}}$ in the model can be obtained consistently and efficiently using equation-by-equation OLS. This is due to the fact that the model has identical regressors in each equation (the excess returns of the market portfolio), as outlined in more detail for instance in Greene (2012).
Equation-by-equation OLS thus also yields a suitable set of residuals $\hat{\varepsilon}_{it}$ to estimate the error covariance matrix $\boldsymbol{S}$, with each element of the matrix given by:

$$\hat{\sigma}_{\varepsilon_{ij}} = \frac{1}{T-2}\sum_{t=1}^{T} \hat{\varepsilon}_{it}\hat{\varepsilon}_{jt} \tag{39}$$

Further, since the excess returns of the market portfolio are assumed i.i.d., a consistent and unbiased estimator of their variance σ_m^2 is the sample variance:

$$\hat{\sigma}_m^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_{mt} - \bar{r}_m)^2 \tag{40}$$

Then, each element of the covariance matrix of excess returns $\boldsymbol{\Sigma}$ can be estimated simply as:

$$\hat{\sigma}_{ij}^2 = \hat{\beta}_i\hat{\beta}_j\hat{\sigma}_m^2 + \hat{\sigma}_{\varepsilon_{ij}} \tag{41}$$

From the first stage, it is thus possible to obtain the consistent estimates $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\Sigma}}$.
To obtain a consistent estimate of the market risk premium μ_m, Fama and MacBeth (1973) use a cross-sectional regression as the second stage, where the obtained estimates $\hat{\boldsymbol{\beta}}$ from the first stage are regressed on the average excess returns of the assets:

$$\bar{r}_i = \hat{\beta}_i\mu_m + \bar{\varepsilon}_i \tag{42}$$

With this estimate $\hat{\mu}_m$, the estimated prior distribution of expected excess returns can be expressed as:

$$\boldsymbol{\mu} \sim N\left(\hat{\boldsymbol{\beta}}\hat{\mu}_m, \tfrac{1}{T}\hat{\boldsymbol{\Sigma}}\right) \tag{43}$$

These estimates can then be used directly in the classical Black-Litterman model as $\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\beta}}\hat{\mu}_m$ and $\hat{\boldsymbol{\Psi}} = \frac{1}{T}\hat{\boldsymbol{\Sigma}}$, where they are combined consistently with the investors subjective views as outlined in Section 3.1.

4.2 Estimation of the Flexible Model

For the flexible model, the following parameterisation was derived in Section 3.5 (see Equation (36)):

$$\boldsymbol{\mu} \sim N\left(\boldsymbol{\beta}\mu_m, \boldsymbol{A} + \tfrac{1}{T}\boldsymbol{\Sigma}\right),$$

Compared to the classical approach, there is one additional parameter matrix to be estimated: $\hat{\boldsymbol{A}}$, an estimate of the covariance matrix of the random intercepts. Therefore, and due to the more complex correlation structure, the estimation process requires additional steps: To estimate the data model outlined in Equation (31), we use a stratified heteroscedastic one-way error components model, which has been studied in the literature by Mazodier and Trognon (1978), Baltagi and Griffin (1988), Phillips (2003) and Hsiao (2003), among others. This section will proceed step-by-step to develop the estimators for the parameters involved, citing relevant literature for mathematical proofs and additional details.

First Stage In the first stage of the regression procedure, a simple transformation of the data allows to compute $\hat{\boldsymbol{S}}$, a consistent estimate of the error covariance matrix. As shown in Equation (34), the sample average of excess returns still contains the random intercept. Then, using demeaned equation-by-equation OLS yields a usable set of residuals to compute $\hat{\boldsymbol{S}}$:

$$r_{it} - \bar{r}_i = (\beta_i r_{mt} + \alpha_i + \varepsilon_{it}) - (\beta_i \bar{r}_m + \alpha_i + \bar{\varepsilon}_i) \tag{44}$$

$$= \beta_i (r_{mt} - \bar{r}_m) + (\varepsilon_{it} - \bar{\varepsilon}_i) \tag{45}$$

$$\dot{r}_{it} = \beta_i \dot{r}_{mt} + \dot{\varepsilon}_{it} \tag{46}$$

Where the last equation uses the definition of a demeaned variable as $\dot{x}_{it} = (x_{it} - \bar{x}_i)$. As the residuals of this estimation are purged of the random intercepts, each element of $\hat{\boldsymbol{S}}$ can be computed as (see, for instance Hsiao (2003), Baltagi (2005) or Greene (2012)):

$$\hat{\sigma}_{\varepsilon_{ij}} = \frac{1}{T-2} \sum_{t=1}^{T} \hat{\dot{\varepsilon}}_{it} \hat{\dot{\varepsilon}}_{jt} \tag{47}$$

Second Stage To estimate $\hat{\boldsymbol{A}}$, we require sample stratification. Too see why, consider first the case without stratification:
For the pooled model as in Equation (31), the error covariance matrix is comprised of both the covariance of random intercepts and the covariance of the error terms, as shown above in Equation (33). The pooled model can be written more concisely as:

$$r_{it} = \beta_i r_{mt} + \nu_{it} \quad \text{with} \quad \nu_{it} = \alpha_i + \varepsilon_{it} \tag{48}$$

Then, $\text{Var}[\nu_{it}] = \sigma^2_{\varepsilon_i} + \sigma^2_{\alpha_i}$. Using these residuals and the estimate $\hat{\sigma}_{\varepsilon_{ij}}$ from the first stage (Equation (47)), Baltagi and Griffin (1988) propose to obtain estimates for the variance of the random intercepts as:

$$\hat{\sigma}^2_{\alpha_i} = \left[\frac{1}{T-2} \sum_{t=1}^{T} \hat{\nu}^2_{it} \right] - \hat{\sigma}^2_{\varepsilon_i} \tag{49}$$

However, Phillips (2003) and Hsiao (2003) show that this estimator suffers from the incidental parameter problem defined by Neyman and Scott (1948). If the variance of α_i changes with every i in an unpredictable way, then there is actually only one draw from the distribution generating α_i, and it is impossible to estimate the variance $\sigma^2_{\alpha_i}$ from this single draw. The estimate converges to $\hat{\alpha}^2_i$ instead of $\hat{\sigma}^2_{\alpha_i}$.

We thus follow the propositions of Mazodier and Trognon (1978) and Phillips (2003) and use sample stratification to solve the incidental parameter problem. In a stratified heteroscedastic one-way error components model, the variances of the random intercepts are modeled as being the same across a cross-sectional subgroup of securities. Each of these groups comprises a stratum and the associated random intercepts are denoted as $\alpha_s^{\{g\}}$ (where s indicates individual securities) with variances $\sigma^2_{\alpha\{g\}}$ for each group $g = 1, \ldots, G$. As a consequence, N_g observations of $\alpha_s^{\{g\}}$ with the same variance are now available, where N_g denotes the number of assets in the gth stratum. This permits the estimation of the variance from N_g observations, instead of one. Intuitively, as a realization of the same random variable is observed for every member of the group, it's variance can be estimated and $\sigma^2_{\alpha\{g\}}$ is no longer incidental.

Note that in the area of asset allocation, the decision maker is interested in allocating wealth across various groups of assets. These groups are represented by market indices, which actually provide natural strata. Thus, in an asset allocation problem, the assets of interest are indices, as indicated by the subscript i throughout the paper. Only here, to estimate $\hat{\sigma}^2_{\alpha_i}$, which suffers from the incidental parameters problem, it is necessary to use securities-level data. Our groups are then the market indices, and we use the index i instead of g to indicate market indices, and N_i is the number of securities in the ith index. We aim at estimating $\sigma^2_{\alpha\{i\}}$, the variance of random intercepts in the ith index.

To obtain an estimate of $\hat{\sigma}^2_{\alpha\{i\}}$, Mazodier and Trognon (1978) propose to apply the following two-stage procedure for each group (or index in our case) individually:

Stage 2.1 First, estimate the equivalent to the pooled model in Equation (48) with equation-by-equation OLS, but for each index $i = 1, \ldots, N$:

$$r_{st}^{\{i\}} = \beta_s^{\{i\}} r_{mt} + \nu_{st}^{\{i\}} \quad \text{with} \quad \nu_{st}^{\{i\}} = \alpha_s^{\{i\}} + \varepsilon_{st}^{\{i\}} \tag{50}$$

$$\text{where} \quad \alpha_s^{\{i\}} \sim N\left(0, \sigma^2_{\alpha\{i\}}\right) \quad \text{and} \quad \varepsilon_{st}^{\{i\}} \sim N\left(0, \sigma^2_{\varepsilon_s^{\{i\}}}\right),$$

Note that $\{i\}$ indicates that a security s belongs to market index i. As before, $\text{Var}[\nu_{st}^{\{i\}}] = \sigma^2_{\varepsilon_s^{\{i\}}} + \sigma^2_{\alpha\{i\}}$, but again, the key difference to the model in Equation (48) and the associated estimate in Equation (49) is that within market index i, the variance of the intercepts is constant and does not change with each security s.

Stage 2.2 Second, estimate for each index $i = 1, \ldots, N$ the demanded model as in Equation (44) with equation-by-equation OLS, to obtain a set of residuals purged of the group effects:

$$\dot{r}_{st}^{\{i\}} = \beta_s^{\{i\}} \dot{r}_{mt} + \dot{\varepsilon}_{st}^{\{i\}} \tag{51}$$

Here, $\text{Var}[\dot{\varepsilon}_{st}^{\{i\}}] = \sigma^2_{\varepsilon_s^{\{i\}}}$. These two sets of residuals can then be combined to obtain the estimates for $\sigma^2_{\alpha\{i\}}$, the variance of the intercepts of each index (Mazodier and Trognon, 1978; Phillips, 2003; Greene, 2012):

$$\hat{\sigma}^2_{\alpha\{i\}} = \frac{1}{N_i}\sum_{s=1}^{N_i}\left[\frac{\sum_{t=1}^{T}\left[\hat{\nu}_{st}^{\{i\}}\right]^2}{T-2} - \frac{\sum_{t=1}^{T}\left[\hat{\varepsilon}_{st}^{\{i\}}\right]^2}{T-2}\right] = \frac{1}{N_i}\sum_{s=1}^{N_i}\widehat{\left[\alpha_s^{\{i\}}\right]^2} \tag{52}$$

The last term in Equation (52) indicates that the incidental parameter problem is solved - the estimator of the variance component for the group effect is a sum of squared residuals, along the cross-section of securities in the index.

This theoretical result provides an intuitive interpretation of the uncertainty that we measure: Uncertainty about the equilibrium model is derived from the average of the squared Jensen's Alphas of the group. The better the equilibrium model explains the assets' returns, the smaller Jensen's Alpha should be and very large deviations from zero indicate a problem with the equilibrium model. This seems a more reasonable approach than to directly assume that Jensen's Alpha is zero, as is done in the classical approach, or to simply ask the investor to specify some confidence parameter τ.

If the investor is not interested in asset allocation but in portfolio allocation, where the decision making is on the level of individual securities, the procedure of Phillips (2003) is a suitable alternative. His model selects the number of strata, as well as the classification of each security into the respective strata automatically, and does not require any pre-selection by the investor. While the estimated $\hat{\sigma}^2_{\alpha_i}$ would be equal for all securities being classified into the same strata, the model maintains more flexibility than the classical Black-Litterman model.

Third Stage With both $\hat{\sigma}^2_{\alpha_i}$ and $\hat{\sigma}^2_{\varepsilon_i}$ obtained as outlined before, feasible generalized least squares is used in the third stage to obtain $\hat{\boldsymbol{\beta}}$ consistently and efficiently. As outlined in Greene (2012), this amounts to estimating the following partially demanded model using equation-by-equation OLS:

$$(r_{it} - \hat{\theta}_i\bar{r}_i) = \beta_i(r_{mt} - \hat{\theta}_i\bar{r}_m) + (\nu_{it} - \hat{\theta}_i\bar{\nu}_i) \tag{53}$$

$$\text{with} \quad \hat{\theta}_i = 1 - \frac{\hat{\sigma}_{\varepsilon_i}}{\sqrt{\hat{\sigma}^2_{\varepsilon_i} + T\hat{\sigma}^2_{\alpha_i}}}$$

With $\hat{\boldsymbol{\beta}}$, $\hat{\sigma}^2_{\alpha_i}$ and $\hat{\sigma}^2_{\varepsilon_i}$ thus estimated, Equations (40) and (41) can be used to obtain $\hat{\sigma}^2_m$ and $\hat{\boldsymbol{\Sigma}}$.

Fourth Stage The last stage corresponds exactly to the last stage of the classical Fama and MacBeth (1973) procedure: To obtain a consistent estimate of the market risk premium μ_m, they use a cross-sectional regression, where the obtained estimates $\hat{\boldsymbol{\beta}}$ from the previous stage are regressed on the average excess returns of the assets:

$$\bar{r}_i = \hat{\beta}_i \mu_m + \bar{\varepsilon}_i \tag{54}$$

While feasible generalized least squares would be more efficient in this case, consistency through OLS is sufficient here.

Summary The procedure to estimate the flexible model requires four different stages, where each stage is necessary to obtain some parameters: The first stage yields an estimate of the error covariance matrix $\hat{\boldsymbol{S}}$; the second stage uses stratification to consistently estimate $\hat{\sigma}^2_{\alpha_i}$; the third stage allows to obtain $\hat{\boldsymbol{\beta}}$, which in turn yields $\hat{\sigma}^2_m$ and $\hat{\boldsymbol{\Sigma}}$; and finally, the fourth stage again uses a simple cross-sectional regression to obtain $\hat{\mu}_m$.
Note, however, that $\hat{\boldsymbol{A}}$ was not directly obtained in the second stage. While it is theoretically feasible to use estimated intercepts of different groups to obtain off-diagonal elements of $\hat{\boldsymbol{A}}$, this has practical limitations, especially when the groups are not of equal size. We thus use the specification outlined in Section 3.3 and in the last paragraph of Section 3.5, which retains the correlation structure of $\hat{\boldsymbol{\Sigma}}$: First, obtain the estimated correlation matrix following Equation (18) as:

$$\hat{\boldsymbol{\Phi}} = \mathrm{diag}(\hat{\boldsymbol{\sigma}})^{-1}\hat{\boldsymbol{\Sigma}}\mathrm{diag}(\hat{\boldsymbol{\sigma}})^{-1}, \tag{55}$$

where $\hat{\boldsymbol{\sigma}}$ is the square root of the diagonal elements of $\hat{\boldsymbol{\Sigma}}$. Then, use the specification of $\hat{\boldsymbol{\Psi}}$ as in Equation (19):

$$\hat{\boldsymbol{\Psi}} = \mathrm{diag}(\hat{\boldsymbol{\sigma}}_{\boldsymbol{\pi}})\hat{\boldsymbol{\Phi}}\mathrm{diag}(\hat{\boldsymbol{\sigma}}_{\boldsymbol{\pi}}), \tag{56}$$

where each element of the vector $\hat{\boldsymbol{\sigma}}_{\boldsymbol{\pi}}$ is now determined according to Equation (37):

$$\hat{\sigma}_{\pi,i} = \sqrt{\hat{\sigma}^2_{\alpha_i} + \tfrac{1}{T}\hat{\sigma}^2_i}, \tag{57}$$

with $\hat{\sigma}^2_i$ the diagonal elements of $\hat{\boldsymbol{\Sigma}}$.
Then, use $\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\beta}}\hat{\mu}_m$ and $\hat{\boldsymbol{\Psi}}$ as specified just above in the classical Black-Litterman model.

5 Application

In the previous sections, we proposed a slight change to the data model underlying the equilibrium part of the classical Black-Litterman model. This change introduces more flexibility and allows the estimation of equilibrium model uncertainty directly from the data. It alleviates the need for investors to set the value of the parameter τ, which is difficult to interpret and has a large effect on the resulting allocation. The proposed change allows the investor to focus solely on the specification of subjective views, as the market equilibrium model is fully specified by the data. The model thus conforms with the assumption of semi-strong-form market efficiency, in the spirit of the original authors of the Black-Litterman model.

In this section, we apply the flexible model to a regional allocation problem in Europe. We illustrate how the model can be applied, reporting intermediate results of the regression stages and interpreting the main results. We also answer two important questions for investors facing similar allocation problems:

1. Is it possible to improve upon simple market capitalization weighted allocations by incorporating equilibrium model uncertainty?
2. When introducing subjective views, how does the flexible model compare to the classical Black-Litterman model?

While the first question can be studied using historical portfolio characteristics, the second question requires a simulation exercise. We start with a discussion of the data in Section 5.1 and an illustration of the equilibrium estimation in Section 5.2. The historical analysis will be discussed in Section 5.3, and Section 5.4 introduces the simulation exercise. Section 5.5 concludes with a number of robustness tests about equilibrium model estimation.

5.1 Data

The European regional allocation problem is based on the MSCI Europe Total Return Index. According to MSCI Inc., the index captures approximately 85% of the free float-adjusted market capitalization across the European Developed Markets equity universe[6]. It is made up of 15 country level sub-indices (Austria, Belgium, Denmark, Finland, France, Germany, Ireland, Italy, the Netherlands, Norway, Portugal, Spain, Sweden, Switzerland and the UK) and, as this is a market capitalization weighted index, individual country indices enter the MSCI Europe Index according to their market capitalization weights. We are interested in deriving new weights for the individual country indices, that take uncertainty into account and allow the specification of subjective views on the performance of individual countries' stock markets by an investor.

We use monthly total return index data for the MSCI Europe Index, for each regional sub-index and for all constituent securities of the index. Also, market capitalization weights are obtained, to allow the reconstruction of the index at various rebalancing frequencies. Data is downloaded in Euros, currency conversions are conducted using spot rates (i.e. foreign currencies are not hedged) and the risk-free rate is represented by the 1-month compounded

[6]See description by MSCI Inc., accessed October 3rd, 2021 at `https://www.msci.com/documents/10199/f6179af3-b1d1-4df0-8ac9-215451f3ac0a`.

Euribor rate as provided by the ECB. We obtain data for a period starting in December 2000 and ending in July 2021, covering roughly 20 years of historical data. Table 1 reports summary statistics of the indices for the entire sample. Differences in performance, both absolute and risk-adjusted (in terms of the annualized Sharpe ratio), are substantial: While an investment in the Danish stock market yielded annual returns of more than 11% (for a Sharpe ratio of 0.65), the same investment in the Portuguese stock market created barely positive annual returns, at only 0.47% (with a Sharpe ratio of 0.03).

SUMMARY STATISTICS

	MSCI ID	$\bar{r}_i$	σ_i	annualized $\bar{r}_i$	annualized σ_i	Sharpe ratio
MSCI Europe	990500	0.33%	4.44%	4.00%	15.38%	0.26
MSCI Austria	904000	0.55%	6.96%	6.74%	24.11%	0.28
MSCI Belgium	905600	0.30%	5.68%	3.71%	19.66%	0.19
MSCI Denmark	920800	0.90%	5.00%	11.32%	17.31%	0.65
MSCI Finland	924600	0.30%	7.69%	3.69%	26.64%	0.14
MSCI France	925000	0.36%	5.07%	4.45%	17.57%	0.25
MSCI Germany	928000	0.44%	5.92%	5.38%	20.52%	0.26
MSCI Ireland	937200	0.10%	6.11%	1.25%	21.17%	0.06
MSCI Italy	938000	0.12%	5.88%	1.44%	20.37%	0.07
MSCI Netherlands	952800	0.53%	5.17%	6.51%	17.90%	0.36
MSCI Norway	957800	0.66%	6.68%	8.25%	23.15%	0.36
MSCI Portugal	962000	0.04%	5.15%	0.47%	17.85%	0.03
MSCI Spain	972400	0.35%	5.99%	4.29%	20.74%	0.21
MSCI Sweden	975200	0.65%	6.31%	8.08%	21.87%	0.37
MSCI Switzerland	975600	0.51%	3.68%	6.30%	12.75%	0.49
MSCI UK	982600	0.19%	4.20%	2.33%	14.56%	0.16

Notes: Data covers the period 2000-12-31 to 2021-07-31 at a monthly frequency, yielding 248 observations. MSCI ID is the *MSCI index code* as provided by MSCI at `https://www.msci.com/ticker-codes`. $\bar{r}_i$ and σ_i are average monthly returns and volatilities, respectively, with their annualized counterparts reported as well. The Sharpe ratio is defined as $\frac{\mathbb{E}[r_i - r_f]}{\sigma_i}$, where annualized values are used and r_f denotes the risk-free rate of return.

Source: Own Calculation. Data: MSCI Inc.

Table 1: Summary statistics for indices used in the estimation.

5.2 Equilibrium Model Estimation

In this section, the estimation of the equilibrium part of both the Black-Litterman and the flexible model is illustrated. As only the equilibrium part is estimated, views are not expressed and do not enter the model. This allows a direct comparison of model-specifications, without the possibly distorting influence of subjective views. We use data from July 2011 to July 2021, for a lookback window of ten years, to capture a typical business cycle.

In addition to the flexible model, we consider two different specifications of the equilibrium uncertainty parameter τ in the classical Black-Litterman model: $\tau = \frac{1}{T}$, the specification proposed by Rachev et al. (2008) and Meucci (2010) and $\tau = 0.05$, the upper bound of the range suggested in Idzorek (2007). We follow the estimation procedures outlined in Sections 4.1 and 4.2 to estimate the parameters of the classical and the flexible model.

Tables 2 and 3 report intermediate results for the two classical Black-Litterman and the flexible model, respectively. The two key features of the flexible model are captured by a single variable, reported in the column τ_i: First, notice how τ_i is constant across all indices for the two Black-Litterman models, while it varies for indices in the flexible model. This is exactly the flexibility we aimed at introducing, carrying additional information not available in the classical model. Second, for the classical Black-Litterman models, τ_i simply reproduces the investor's specification of τ ($\tau = \frac{1}{T} = 0.008$ and $\tau = 0.05$, respectively). For the flexible model, no such specification was necessary. While the simple average of τ_i over all countries is 0.053, comparable to the specified value of 0.05 in one of the classical models and close the proposed range of Idzorek (2007), the parameters were estimated entirely from the data.

Looking at Table 2 in more detail, column $\hat{\beta}_i$ reports the estimated market-beta coefficients of the first stage Fama and MacBeth (1973) regression, as specified in Equation (38), with associated standard errors in parenthesis. As the estimation is independent of the parameter τ, betas are equivalent for both Black-Litterman models. In the second stage, the market risk premium $\hat{\mu}_m$ is estimated to be 0.707%, which allows the computation of $\hat{\pi}_i$ according to Equation (43). They are again identical for both models, as τ has no influence on this estimation. The parameters $\hat{\sigma}^2_{\pi,i}$ report the uncertainty about equilibrium returns and are the diagonal elements of the hyperparameter $\hat{\mathbf{\Psi}} = \tau\hat{\mathbf{\Sigma}}$, which is discussed in more detail in Section 3.2. Note that $\hat{\mathbf{\Sigma}}$ is estimate according to Equation (41) with a market variance $\hat{\sigma}^2_m$ of 0.0015. This column is different for the two Black-Litterman models: the uncertainty is much larger for the model with $\tau = 0.05$ than for the model with $\tau = \frac{1}{T} = 0.008$, a fact that will also ultimately affect the estimated country weights. Finally, Table 2 reports the individual τ_i, which are the same for each country in the classical Black-Litterman model, but are obviously different for the two models.

Table 3 reports the same intermediate results for the flexible model, however, as there are two more stages involved, additional parameters are reported. These are the intermediate results of the second stage and highlight the differences between the classical Black-Litterman model and the more flexible model. First, N_i reports the number of underlying securities used in the estimation of the stratified second stage. We use securities only if there are at least 100 consecutive historical observations available. Next, the distinguishing feature of the flexible model is reported, the estimated variances of the intercepts $\hat{\sigma}^2_{\alpha_i}$ from the stratified second

ESTIMATED PARAMETERS: CLASSICAL BLACK-LITTERMAN MODELS

	BL ($\tau = \frac{1}{T}$)					BL ($\tau = 0.05$)				
	$\hat{\beta}_i$	(sd)	$\hat{\pi}_i$	$\hat{\sigma}^2_{\pi,i}$ $\cdot 10^{-4}$	τ_i	$\hat{\beta}_i$	(sd)	$\hat{\pi}_i$	$\hat{\sigma}^2_{\pi,i}$ $\cdot 10^{-4}$	τ_i
MSCI Austria	1.43	(0.10)	1.01%	0.400	0.008	1.43	(0.10)	1.01%	2.422	0.050
MSCI Belgium	1.05	(0.07)	0.74%	0.208	0.008	1.05	(0.07)	0.74%	1.258	0.050
MSCI Denmark	0.72	(0.07)	0.51%	0.147	0.008	0.72	(0.07)	0.51%	0.890	0.050
MSCI Finland	0.96	(0.07)	0.68%	0.186	0.008	0.96	(0.07)	0.68%	1.128	0.050
MSCI France	1.14	(0.03)	0.81%	0.180	0.008	1.14	(0.03)	0.81%	1.088	0.050
MSCI Germany	1.22	(0.04)	0.86%	0.217	0.008	1.22	(0.04)	0.86%	1.310	0.050
MSCI Ireland	0.90	(0.08)	0.64%	0.208	0.008	0.90	(0.08)	0.64%	1.260	0.050
MSCI Italy	1.32	(0.08)	0.93%	0.312	0.008	1.32	(0.08)	0.93%	1.887	0.050
MSCI Netherlands	0.96	(0.04)	0.68%	0.139	0.008	0.96	(0.04)	0.68%	0.844	0.050
MSCI Norway	1.12	(0.08)	0.79%	0.249	0.008	1.12	(0.08)	0.79%	1.508	0.050
MSCI Portugal	0.79	(0.09)	0.56%	0.193	0.008	0.79	(0.09)	0.56%	1.170	0.050
MSCI Spain	1.21	(0.08)	0.85%	0.286	0.008	1.21	(0.08)	0.85%	1.730	0.050
MSCI Sweden	1.05	(0.05)	0.74%	0.174	0.008	1.05	(0.05)	0.74%	1.054	0.050
MSCI Switzerland	0.69	(0.04)	0.48%	0.086	0.008	0.69	(0.04)	0.48%	0.520	0.050
MSCI UK	0.95	(0.03)	0.67%	0.130	0.008	0.95	(0.03)	0.67%	0.789	0.050

Notes: Data used to estimate models covers the period 2011-07-31 to 2021-07-31 at a monthly frequency, yielding 121 observations. $\hat{\beta}_i$ are the estimated market-beta coefficients in the first stage of the regression model (including standard errors in parenthesis). $\hat{\sigma}^2_{\pi,i}$ are the diagonal elements of $\mathbf{\Psi} = \tau\mathbf{\Sigma}$, where each entry is scaled for readability and multiplied by 10^4. τ_i are the corresponding proportionality coefficients (constant across different i in the classical case). $\hat{\pi}_i = \hat{\beta}_i\hat{\mu}_m$ are the expected equilibrium returns, where the market risk premium is estimated in the second stage as $\hat{\mu}_m = 0.707\%$.

Source: Own Calculation. Data: MSCI Inc.

Table 2: Intermediate results for the estimation of the classical Black-Litterman models.

stage. They are obtained according to Equation (52) and measure the uncertainty about the equilibrium model for each index. For instance, they indicate that the CAPM does not explain the stock markets of Denmark and Ireland very well, while the fit is better for Austria, Belgium or Norway. Finally, $\frac{1}{T}\hat{\sigma}^2_i$ measure the estimation error and correspond to the diagonal elements of $\frac{1}{T}\hat{\mathbf{\Sigma}}$. They are very similar[7] to $\hat{\sigma}^2_{\pi,i}$ for the classical Black-Litterman model with $\tau = \frac{1}{T}$. The remaining parameters in Table 3 are familiar from Table 2. The market-betas $\hat{\beta}_i$ are slightly different, as they are estimated in the third stage using FGLS instead of OLS. This also leads to a slightly different estimate of the market risk premium $\hat{\mu}_m$ of 0.709%, and in turn to new estimates for $\hat{\pi}_i$. Note, however, that these differences are very small. More relevant are the differences in the estimated uncertainty about equilibrium returns $\hat{\sigma}^2_{\pi,i}$. Here, this uncertainty is the sum of $\hat{\sigma}^2_{\alpha_i}$ and $\frac{1}{T}\hat{\sigma}^2_i$ (as defined in Equation (36)), the uncertainty about the equilibrium model and the estimation error. Note that it does not depend on any specified parameter τ, but

[7] They are not exactly identical since Equation (41) is used to estimate $\hat{\mathbf{\Sigma}}$, which depends on the estimated $\hat{\beta}_i$.

is directly obtained from the data. Finally, as mentioned above, the flexibility of the approach is evident from the parameters τ_i, which are, in contrast to the classical Black-Litterman model, now distinct for each country.

Estimated Parameters: Flexible Model (FL)

	N_i	$\hat{\sigma}^2_{\alpha_i} \cdot 10^4$	$\frac{1}{T}\hat{\sigma}^2_i \cdot 10^4$	$\hat{\beta}_i$	(sd)	$\hat{\pi}_i$	$\hat{\sigma}^2_{\pi,i} \cdot 10^4$	τ_i
MSCI Austria	5	0.319	0.396	1.42	(0.09)	1.01%	0.715	0.015
MSCI Belgium	11	0.442	0.207	1.05	(0.07)	0.74%	0.649	0.026
MSCI Denmark	17	2.797	0.147	0.72	(0.07)	0.51%	2.944	0.166
MSCI Finland	11	0.959	0.186	0.96	(0.07)	0.68%	1.145	0.051
MSCI France	68	0.713	0.180	1.14	(0.03)	0.81%	0.892	0.041
MSCI Germany	48	0.993	0.216	1.22	(0.04)	0.87%	1.210	0.046
MSCI Ireland	5	1.579	0.208	0.90	(0.08)	0.64%	1.786	0.071
MSCI Italy	19	0.928	0.311	1.32	(0.08)	0.93%	1.239	0.033
MSCI Netherlands	15	0.871	0.139	0.96	(0.04)	0.68%	1.010	0.060
MSCI Norway	9	0.387	0.248	1.12	(0.08)	0.79%	0.635	0.021
MSCI Portugal	4	0.668	0.192	0.78	(0.09)	0.56%	0.860	0.037
MSCI Spain	16	0.621	0.284	1.20	(0.08)	0.85%	0.905	0.026
MSCI Sweden	34	0.658	0.174	1.05	(0.05)	0.75%	0.832	0.040
MSCI Switzerland	39	1.002	0.086	0.69	(0.04)	0.49%	1.088	0.105
MSCI UK	83	0.788	0.130	0.95	(0.03)	0.67%	0.918	0.058

Notes: Data used to estimate the model covers the period 2011-07-31 to 2021-07-31 at a monthly frequency, yielding 121 observations. N_i describes the number of constituents in each index used to estimate the second stage. Constituents are excluded if less than 100 consecutive observations are available. $\hat{\sigma}^2_{\alpha_i}$ is the estimated variance of the random intercept, obtained in the second stage (scaled by 10^4 for readability). $\frac{1}{T}\hat{\sigma}^2_i$ measures the estimation error in the model (scaled by 10^4 for readability). $\hat{\beta}_i$ are the estimated market-beta coefficients in the third stage of the regression model. $\hat{\sigma}^2_{\pi,i}$ are the diagonal elements of $\mathbf{\Psi}$, exactly the sum of $\hat{\sigma}^2_{\alpha_i}$ and $\frac{1}{T}\hat{\sigma}^2_i$ (scaled by 10^4 for readability). τ_i are the corresponding proportionality coefficients (no longer constant, a key feature of the flexible model). $\hat{\pi}_i = \hat{\beta}_i\hat{\mu}_m$ are the expected equilibrium returns, where the market risk premium is estimated in the fourth stage as $\hat{\mu}_m = 0.709\%$.

Source: Own Calculation. Data: MSCI Inc.

Table 3: Intermediate results for the estimation of the flexible model.

5.3 Equilibrium Model Characteristics

We can now use the estimated parameters from Section 5.2 to derive the new allocation weights for each model. Since we do not specify any views in this part of the application, deriving the adjusted parameters $\tilde{\boldsymbol{\mu}}$ and $\tilde{\mathbf{\Sigma}}$ is simple. Equations (11) and (12) reduce to $\tilde{\boldsymbol{\mu}} = \hat{\boldsymbol{\pi}}$ and $\tilde{\mathbf{\Sigma}} = \hat{\mathbf{\Sigma}} + \hat{\mathbf{\Psi}}$, meaning that the posterior distribution can be characterised with the parameters estimated in Section 5.2. As discussed in Section 3.1, these new parameters then serve as the basis for classical mean-variance optimization, yielding the optimal allocation weights. In our optimization routine, we maximize the expected return while keeping the expected risk at $\hat{\sigma}^2_m$, the same risk that the MSCI Europe exhibited in the sample period. That is, we construct a regional allocation that exhibits the same risk as the market capitalization weighted benchmark, but aim at higher returns. To avoid negative weights on individual countries, we always and in all specifications impose a long-only constraint in the optimizations that follow.

Tables 4 and 5 report the results of the optimization exercise. In addition to the previously used models, results for simple market capitalization weighting are reported for comparison. From the tables, two results stand out: First, Table 4 reports the *Herfindahl–Hirschman Index* HHI, a measure of portfolio diversification, along with the portfolio weights. It is defined as the sum of squared allocation weights, and lies between $\frac{1}{N} = 6.66\%$ (perfect diversification) and 100% (allocation concentrated in a single index). Apart from simple capitalization weighting, the flexible model clearly achieves the best diversification. Second, Table 5 reports the historical performances of the same models. In the last column, a direct comparison between the flexible model and the most common specification of the classical Black-Litterman model (where $\tau = \frac{1}{T}$) is reported. Here, the flexible model achieves a statistically significant outperformance of 0.45% p.a., measured by Jensen's Alpha. This is a strong indication that the flexible model makes more efficient use of the available data in estimating the equilibrium allocation.

Looking at Table 4 in more detail, some notable differences in the allocations arise. On the one hand, differences between the two Black-Litterman specification can be attributed to the added uncertainty incorporated into updated parameters. On the other hand, the deviations in the flexible model reflect the individually estimated uncertainties, for instance for Denmark, where the perceived higher risk translates into a lower allocation, or Norway, where the reverse is true. To evaluate the different approaches, we again highlight the Herfindahl–Hirschman Index measuring portfolio diversification. The market capitalization weighted allocation exhibits the best diversification, very closely followed by the flexible model. Diversification is clearly reduced in both Black-Litterman specifications. To put these results into perspective, for a simple mean-variance optimization that uses historical average returns and covariance parameters (without relying on an equilibrium approach), the resulting portfolio would allocate 79.6% to Denmark and the remaining 20.4% to the Netherlands, for a HHI of 67.6%. This clearly highlights the problems of simple mean variance optimization and the resulting highly concentrated portfolios.

To further evaluate the allocations, we run historical backtests using all available data, from December 2000 to July 2021, a total of 248 observations. We choose a monthly rebalancing frequency and a sliding window of 121 observations, such that each optimization is based upon

Allocation Weights

	Market Capitalization	BL ($\tau = \frac{1}{T}$)	BL ($\tau = 0.05$)	FL
MSCI Austria	0.30%	-	-	-
MSCI Belgium	1.42%	1.01%	0.06%	3.07%
MSCI Denmark	4.12%	2.98%	3.63%	0.07%
MSCI Finland	1.70%	1.85%	2.42%	2.37%
MSCI France	17.77%	14.33%	13.08%	10.69%
MSCI Germany	14.37%	13.02%	10.79%	8.41%
MSCI Ireland	1.09%	0.97%	0.99%	1.36%
MSCI Italy	3.78%	3.52%	2.73%	1.79%
MSCI Netherlands	6.92%	6.66%	7.65%	6.78%
MSCI Norway	0.94%	1.07%	0.79%	2.22%
MSCI Portugal	0.28%	0.40%	0.89%	3.59%
MSCI Spain	3.65%	4.32%	4.65%	5.71%
MSCI Sweden	5.95%	5.18%	4.67%	8.53%
MSCI Switzerland	15.41%	15.51%	17.70%	18.78%
MSCI UK	22.30%	29.18%	29.94%	26.62%
HHI	13.92%	15.85%	16.28%	14.35%

Notes: Data used to estimate models covers the period 2011-07-31 to 2021-07-31 at a monthly frequency, yielding 121 observations. Market Capitalization weights are observed directly and reported for comparison. Models denoted as *BL* are estimated Black-Litterman models with the indicated value for τ. *FL* indicates the proposed flexible model. The HHI is the Herfindahl-Hirschman Index, a measure of portfolio diversification. It is defined as HHI $= \sum_{i=1}^{N} w_i^2$. A smaller value indicates better diversification. It can range from $\frac{1}{N} = 6.66\%$ (equal weighted allocation) to 100% (allocation concentrated in a single index).

Source: Own Calculation. Data: MSCI Inc.

Table 4: Comparison of regional allocation weights from different models.

ten years of returns data.[8] The optimization for the last sliding window, from July 2011 to July 2021 is thus exactly the optimization just presented.

With a total of 248 monthly observations, we obtain 127 sliding windows of appropriate length. For each window, we estimate the models to derive optimal allocation weights and compute performance time series from these optimal weights. The resulting time series cover the period from December 2010 (first full sliding window) to July 2021 (last sliding window, details about estimation for this allocation reported in Section 5.2) and are the basis for calculating the performance measures in Table 5. The relative measures in the lower part of the table are computed against the MSCI Europe Total Return Index, except for the last column, where the two models indicated are directly compared.

Looking at absolute measures first, the flexible model performed best, both in terms of

[8]We test these settings in Section 5.5.

Performance Characteristics

	Market Capitalization	BL ($\tau = \frac{1}{T}$)	BL ($\tau = 0.05$)	FL	FL vs. BL ($\tau = \frac{1}{T}$)
Return p.a.	9.19%	9.02%	9.13%	9.34%	
Volatility p.a.	13.40%	13.47%	13.24%	13.25%	
Sharpe Ratio	0.69	0.67	0.69	0.71	
relative Return p.a.	0.01%	−0.15%	−0.05%	0.15%	0.30%
Tracking Error p.a.	0.04%	0.44%	0.58%	0.84%	0.64%
Information Ratio	0.29	−0.34	−0.09	0.18	0.47
Regression Beta	1.00**	1.00**	0.99**	0.99**	0.98**
Std. Dev. Beta	0.000	0.003	0.005	0.006	0.005
Regression Alpha p.a.	0.02%	−0.19%	0.07%	0.27%	0.45%**
T-Stat. of Alpha	1.20	−1.39	0.37	1.05	2.33
p-Val. of Alpha	0.229	0.166	0.714	0.296	0.020

Notes: Data used to estimate backtests covers the period 2000-12-31 to 2021-07-31 at a monthly frequency, yielding 248 observations. For each period, a sliding window of data is used to estimate the optimal allocation, each window covering $T = 121$ observations, with a monthly rebalancing frequency. Optimal allocations are thus available for the period 2010-12-31 to 2021-07-31 for a total of 127 observations of the performance at a monthly frequency. Results report characteristics of these time series. Relative measures are computed against the MSCI Europe Total Return Index (EUR), except for the last column, where they are computed against the Black-Litterman model with $\tau = \frac{1}{T}$. The performed regressions are specified as in Equation (38).

* Significant at the 10 percent level.

** Significant at the 5 percent level.

Source: Own Calculation. Data: MSCI Inc.

Table 5: Comparison of performance characteristics from backtests of different models.

absolute returns with 9.34% against 9.02% and 9.13% for the Black-Litterman specifications, as well as in terms of risk adjusted performance, with a Sharpe ratio of 0.71, against 0.67 and 0.69 for the Black-Litterman specifications. The flexible model also compares favourably against the market capitalization weighted allocation.

For the relative performance measures[9], a clear picture emerges. While both Black-Litterman specifications exhibit negative relative returns, the flexible model beats the benchmark by about 15 basis points annually. This is corroborated by risk-adjusted regression results, as the flexible model has an annualized alpha of 0.27% (although it is not statistically significant). The Black-Litterman specifications have lower or negative alphas: The specification with $\tau = \frac{1}{T}$ has a negative alpha, the one with $\tau = 0.05$ an alpha close to zero. When directly comparing the flexible model with the commonly used specification of $\tau = \frac{1}{T}$ for the Black-Litterman model, as is done in the last column of the table, we obtain the previously mentioned statistically

[9]Relative measures for the market capitalization weights (first column) only measure differences in performance arising from the monthly rebalancing frequency (the MSCI Europe Index has a quarterly rebalancing frequency). As such, relative measures are very close to zero.

significant outperformance of the flexible model, with an annualized alpha of 0.45%.

The flexible model departs from the classical Black-Litterman model in two important ways: First, uncertainty about the equilibrium model is estimated from the data and not set subjectively by the investor. Second, the uncertainty is estimated flexibly, separate for each index in the allocation. An interesting question then is the relative importance of these two innovations in explaining the measured outperformance. In the classical Black-Litterman model, specifying $\tau = \frac{1}{T}$ serves as a starting point. Does the flexible model outperform due to a better estimate of τ, or due to the flexibility? To investigate this issue, we repeat the historical analysis for a model with the following properties: For each period, the individual τ_i are estimated as in the flexible model. We then take their market-capitalization-weighted average and use this estimate as the parameter τ in a classical Black-Litterman model. Formally, for each period, the value of the parameter is defined as:

$$\tau^t = \overline{\tau_i}^t = \sum_{i=1}^{n} \omega_{m,i}^t \tau_i^t, \tag{58}$$

where τ_i^t are the individually estimated uncertainty parameters from the flexible model for period t, and $\omega_{m,i}^t$ are the contemporaneous market capitalization weights. The value of this parameter over time is depicted in Figure 1, highlighting its dynamic nature. Table 6 reports

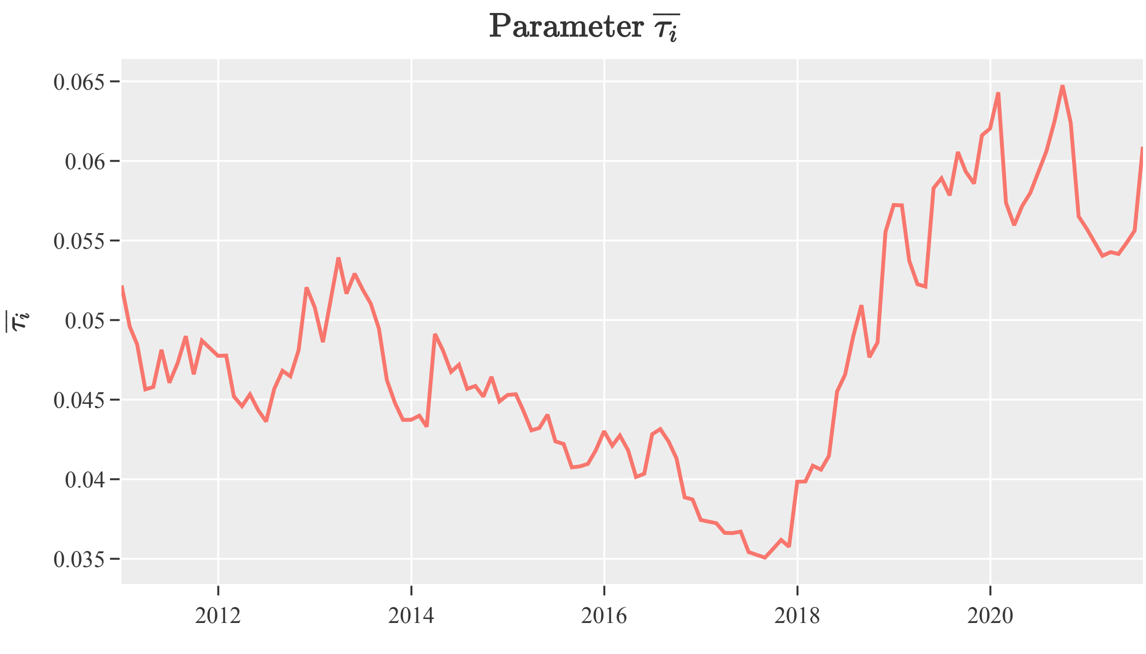

Figure 1: Weighted average of estimated individual τ_i over time.

the results of the historical analysis. The first column reports the performance measures of this newly specified model with estimates of τ^t taken from the flexible model. Columns two and three then use these results to attribute the positive alpha of the flexible model of 0.45% to either the estimated level of τ^t, or the flexibility of the model (that is, individual τ_i^t for each index in the allocation). As reported in the second column, a statistically significant 0.28% of the alpha is due to the estimated level of τ^t (roughly 60% of the alpha). Another 0.18% (the remaining 40% of alpha) is attributable to the flexibility of the model. Although not statistically significant, the increased flexibility still adds to the overall performance of the model.

Note that the stand-alone results of the newly specified model are actually similar to the results of the classical Black-Litterman model with $\tau = 0.05$, reported in Table 5. This is no coincidence, as the estimated τ^t hovers around 0.05 over time in Figure 1, with an actual cross-temporal average of 0.048. Note however, that previously we had set the value $\tau = 0.05$ blindly, simply because it was the upper bound of the range suggested by Idzorek (2007). It was a lucky guess, so to speak, and highlights again the importance of our contribution of allowing to estimate the parameter directly from the data.

To conclude, we find indication that incorporating uncertainty into the equilibrium model can improve upon simple market capitalization weights. However, this is only the case when using the proposed flexible model, which achieves a statistically significant alpha compared to the most common Black-Litterman specification. This alpha is to approximately 60% attributable to the estimation of model risk from the data, and to 40% to the increased flexibility of the model.

5.4 Views

In the previous section, we focused the investigation on the equilibrium part of the models. While this allows for a direct comparison of different models, absent any subjective views, the main advantage of the Bayesian framework is the possibility of incorporating subjective views. With the encouraging results from studying the flexible model's equilibrium model properties, we now turn to investigating it's properties in the presence of subjective views.

Objectively evaluating different models under subjective views is a challenging exercise, as the performances of the models heavily depend on the quality of the views. The only difference between the models we aim to evaluate is the value of the hyperparameter $\mathbf{\Psi}$, controlling the mixing of the two sources of information (equilibrium model and views). As such, it is theoretically possible to always engineer subjective views that favour one model over the other. For instance, if one model attributes less uncertainty to the equilibrium model than another, it will outperform if the subjective views turn out to be wrong, but will perform worse if the information was good. The same is true for the flexible model: As it attributes uncertainty on

Outperformance Attribution

	BL ($\tau^t = \overline{\tau_i}^t$)	BL ($\tau^t = \overline{\tau_i}^t$) vs. BL ($\tau = \frac{1}{T}$)	FL vs. BL ($\tau^t = \overline{\tau_i}^t$)
Return p.a.	9.14%		
Volatility p.a.	13.21%		
Sharpe Ratio	0.70		
relative Return p.a.	−0.04%	0.11%	0.19%
Tracking Error p.a.	0.60%	0.45%	0.53%
Information Ratio	−0.07	0.24	0.36
Regression Beta	0.99**	0.98**	1.00**
Std. Dev. Beta	0.006	0.005	0.004
Regression Alpha p.a.	0.09%	0.28%**	0.18%
T-Stat. of Alpha	0.49	2.07	1.02
p-Val. of Alpha	0.621	0.039	0.310

Notes: Data used to estimate backtests covers the period 2000-12-31 to 2021-07-31 at a monthly frequency, yielding 248 observations. For each period, a sliding window of data is used to estimate the optimal allocation, each window covering $T = 121$ observations, with a monthly rebalancing frequency. Optimal allocations are thus available for the period 2010-12-31 to 2021-07-31 for a total of 127 observations of the performance at a monthly frequency. Results report characteristics of these time series. The model indicated by BL ($\tau^t = \overline{\tau_i}^t$) uses the weighted average of estimated τ_i from the flexible model in each period to then use classical Black-Litterman with that estimate as τ. Relative measures are computed against the MSCI Europe Total Return Index (EUR) in the first column, and against the models indicated in the header for subsequent columns. Details about these other models can be obtained from Table 5. The performed regressions are specified as in Equation (38).

* Significant at the 10 percent level.

** Significant at the 5 percent level.

Source: Own Calculation. Data: MSCI Inc.

Table 6: Attribution of the flexible models' outperformance to estimation of τ and model flexibility.

the level of individual indices, it would be trivial to engineer views where the model performs very good by assigning ex post correct views for assets with high uncertainty and ex post incorrect views for assets with low uncertainty.

To achieve at least some objectivity, we rely on and extend the work by Gofman and Manela (2012). In their paper, the authors introduce a simple procedure to randomly generate views that carry a signal, albeit hidden behind a controllable amount of noise. Views are specified for all assets and are always absolute (i.e. the views matrix $\boldsymbol{P}$ is an $(N\times N)$ identity matrix). Then, the vector of expected returns of the views is generated according to the following equation:

$$\boldsymbol{q} = \boldsymbol{r}_{t+1} + \breve{\sigma}\tilde{\boldsymbol{e}}, \tag{59}$$

$$\text{with} \qquad \tilde{\boldsymbol{e}} \sim N(\mathbf{0}, \tilde{\boldsymbol{\Sigma}}),$$

where $\boldsymbol{r}_{t+1}$ is the vector of next period returns, $\tilde{\boldsymbol{e}}$ is random noise, drawn from a zero centred normal distribution with covariance matrix $\tilde{\boldsymbol{\Sigma}}$, and $\breve{\sigma}$ is the noise parameter, controlling the size of the random disturbances. In this setting, the investors get the ultimate information, a peak at next periods returns, hidden behind a controllable amount of noise. As proposed by Gofman and Manela (2012), we sample 300 realizations of $\tilde{\boldsymbol{e}}$ for each date, and use the same realizations across the different models. Like that, we create 300 different backtest time series, allowing us to compute performance statistics including their standard deviation.

Table 7 reports the results of this simulation exercise, where the most relevant performance measures (with standard errors in parenthesis) are reported for each model at varying levels of the noise factor $\breve{\sigma}$. As can be expected, when the signal is not well hidden, all models lead to substantial positive performance, high Sharpe ratios and very high regression alphas. The private information is successfully incorporated into the allocation and leads to better results. The quality of the results gradually declines with a larger noise factor, and for a very large level of noise, the Black-Litterman specification with the greatest reliance on the equilibrium model exhibits a negative alpha. Comparing the models, note that the flexible model's alpha always lies between the two Black-Litterman specifications. Recall that it is to be expected that the Black-Litterman specification with a larger τ will outperform in this case, as the views enclose a perfect signal of next period returns, and thus relying more on the views and less on the equilibrium will lead to a better performance by construction. As the flexible model exhibits an average τ that is smaller than, but close to, 0.05, the observed performance is within expectations.

Evaluation of Views: Perfect Information

	Return p.a.		Sharpe Ratio		Reg. Alpha p.a.	
	Mean	Std.	Mean	Std.	Mean	Std.
$\breve{\sigma} = 1$						
BL ($\tau = \frac{1}{T}$)	31.13%	(0.70%)	2.30	(0.06)	20.60%	(0.67%)
BL ($\tau = 0.05$)	37.10%	(0.99%)	2.85	(0.09)	27.00%	(0.95%)
FL	34.10%	(1.17%)	2.67	(0.10)	24.41%	(1.11%)
$\breve{\sigma} = 2$						
BL ($\tau = \frac{1}{T}$)	21.29%	(0.90%)	1.58	(0.07)	11.44%	(0.83%)
BL ($\tau = 0.05$)	25.61%	(1.34%)	1.98	(0.11)	16.26%	(1.29%)
FL	23.78%	(1.41%)	1.86	(0.11)	14.75%	(1.32%)
$\breve{\sigma} = 4$						
BL ($\tau = \frac{1}{T}$)	15.15%	(0.92%)	1.09	(0.10)	5.77%	(0.86%)
BL ($\tau = 0.05$)	17.73%	(1.35%)	1.33	(0.21)	8.86%	(1.28%)
FL	16.69%	(1.46%)	1.26	(0.21)	8.10%	(1.36%)
$\breve{\sigma} = 8$						
BL ($\tau = \frac{1}{T}$)	11.92%	(0.92%)	0.89	(0.07)	2.79%	(0.86%)
BL ($\tau = 0.05$)	13.44%	(1.34%)	1.04	(0.11)	4.84%	(1.26%)
FL	12.82%	(1.50%)	1.00	(0.12)	4.44%	(1.40%)
$\breve{\sigma} = 16$						
BL ($\tau = \frac{1}{T}$)	10.31%	(0.93%)	0.77	(0.07)	1.30%	(0.87%)
BL ($\tau = 0.05$)	11.30%	(1.35%)	0.87	(0.11)	2.83%	(1.28%)
FL	10.87%	(1.49%)	0.85	(0.12)	2.60%	(1.38%)
$\breve{\sigma} = 32$						
BL ($\tau = \frac{1}{T}$)	9.50%	(0.92%)	0.71	(0.07)	0.56%	(0.87%)
BL ($\tau = 0.05$)	10.23%	(1.34%)	0.79	(0.11)	1.83%	(1.28%)
FL	9.91%	(1.48%)	0.77	(0.12)	1.69%	(1.38%)

Notes: Reported are averages and standard deviations of running 300 historical tests with randomized views. The same randomization is used for every model. The parameter $\breve{\sigma}$ controls the level of noise by scaling the randomization terms linearly. The signal corresponds to the true next level returns. Relative measures are computed against the MSCI Europe Total Return Index (EUR).

Source: Own Calculation. Data: MSCI Inc.

Table 7: Evaluation of views with perfect information for various models and noise factors.

As a next step, we extend the analysis of Gofman and Manela (2012) by introducing one additional controlling parameter: a bias factor. While in the case of Gofman and Manela (2012) the investor always received the ultimate signal behind the noise, we introduce the possibility that the signal is biased. Let us redefine the vector $\boldsymbol{q}$ as follows:

$$\boldsymbol{q} = \hat{\boldsymbol{\pi}} + \breve{\kappa}\left(\boldsymbol{r}_{t+1} - \hat{\boldsymbol{\pi}}\right) + \breve{\sigma}\tilde{\boldsymbol{e}}, \tag{60}$$

where all definitions are retained from Equation (59). Additionally, $\hat{\boldsymbol{\pi}}$ is the vector of estimated equilibrium returns and $\breve{\kappa}$ is the bias parameter. For $\breve{\kappa} = 1$, the familiar result from Gofman and Manela (2012) and Equation (59) is obtained. However, it is possible to specify $\breve{\kappa} = -1$, in which case the signal received by the investor is "bad": Each view deviates from the equilibrium estimate the same amount as for the perfect signal, but in the opposite direction. As any forecast can be systematically biased, it seems reasonable to investigate how the models would perform under various values for $\breve{\kappa}$.

Table 8 reports the results of the simulation exercise with a value for $\breve{\kappa} = -1$, to evaluate the performance of the models under adverse conditions. Also here, we clearly see the effect of the misinformation of the investor. When in the previous setting, the correct information lead to very good performance results, here we see the opposite effect. Also, for a noise factor smaller than 10, the Black-Litterman specification relying more on the equilibrium model (smaller τ) performs better. When the noise level gets larger, however, this trend slightly reverses, and for very high values of the noise factor, the specification with a higher τ outperforms. Turning to the flexible model, as before, it's performance lies in between the Black-Litterman specifications for each level of the noise factor. This underlines the practical usefulness of the model: As it is impossible to know the quality of a view ex ante, the flexible model will lead to stable results, without the need to specify the uncertainty parameter τ.

Evaluation of Views: Full Bias

	Return p.a.		Sharpe Ratio		Reg. Alpha p.a.	
	Mean	Std.	Mean	Std.	Mean	Std.
$\breve{\sigma} = 1$						
BL ($\tau = \frac{1}{T}$)	-9.08%	(0.49%)	-0.66	(0.04)	-16.64%	(0.46%)
BL ($\tau = 0.05$)	-12.21%	(0.65%)	-0.90	(0.05)	-19.24%	(0.61%)
FL	-11.17%	(0.74%)	-0.83	(0.05)	-18.19%	(0.72%)
$\breve{\sigma} = 2$						
BL ($\tau = \frac{1}{T}$)	-2.30%	(0.69%)	-0.17	(0.05)	-10.38%	(0.66%)
BL ($\tau = 0.05$)	-4.76%	(1.00%)	-0.35	(0.07)	-12.26%	(0.95%)
FL	-4.14%	(1.12%)	-0.31	(0.08)	-11.62%	(1.08%)
$\breve{\sigma} = 4$						
BL ($\tau = \frac{1}{T}$)	2.66%	(0.81%)	0.20	(0.06)	-5.78%	(0.77%)
BL ($\tau = 0.05$)	1.37%	(1.20%)	0.11	(0.09)	-6.49%	(1.15%)
FL	1.63%	(1.38%)	0.13	(0.11)	-6.16%	(1.33%)
$\breve{\sigma} = 8$						
BL ($\tau = \frac{1}{T}$)	5.57%	(0.88%)	0.42	(0.07)	-3.09%	(0.84%)
BL ($\tau = 0.05$)	5.09%	(1.29%)	0.39	(0.10)	-3.01%	(1.24%)
FL	5.15%	(1.45%)	0.40	(0.11)	-2.82%	(1.39%)
$\breve{\sigma} = 16$						
BL ($\tau = \frac{1}{T}$)	7.12%	(0.90%)	0.54	(0.07)	-1.65%	(0.86%)
BL ($\tau = 0.05$)	7.10%	(1.33%)	0.55	(0.10)	-1.11%	(1.27%)
FL	7.02%	(1.48%)	0.54	(0.12)	-1.04%	(1.40%)
$\breve{\sigma} = 32$						
BL ($\tau = \frac{1}{T}$)	7.91%	(0.91%)	0.59	(0.07)	-0.92%	(0.86%)
BL ($\tau = 0.05$)	8.12%	(1.34%)	0.63	(0.11)	-0.15%	(1.28%)
FL	7.97%	(1.47%)	0.62	(0.11)	-0.13%	(1.39%)

Notes: Reported are averages and standard deviations of running 300 historical tests with randomized views. The same randomization is used for every model. The parameter $\breve{\sigma}$ controls the level of noise by scaling the randomization terms linearly. The signal corresponds to the specification in Equation (60), with the bias parameter $\breve{\kappa} = -1$. Relative measures are computed against the MSCI Europe Total Return Index (EUR).

Source: Own Calculation. Data: MSCI Inc.

Table 8: Evaluation of views with a bias of $\breve{\kappa} = -1$ for various models and noise factors.

With views defined as in Equation (60), we can extend the analysis to a larger range for the bias parameter $\breve{\kappa}$. Figure 2 reports the results of simulations with $\breve{\kappa}$ in the range $[-1.5, 1.5]$ graphically. The same results are also depicted in Figure 3, but are more readable, as here the *differences* in Sharpe ratios of the two Black-Litterman models against the flexible model are reported, including their 95% confidence bands arising from 300 historical tests.

Figure 3 illustrated well the quality of the trade-off achieved by the flexible model: Good information ($\breve{\kappa} > 0$) leads to better performance of the Black-Litterman model with a higher

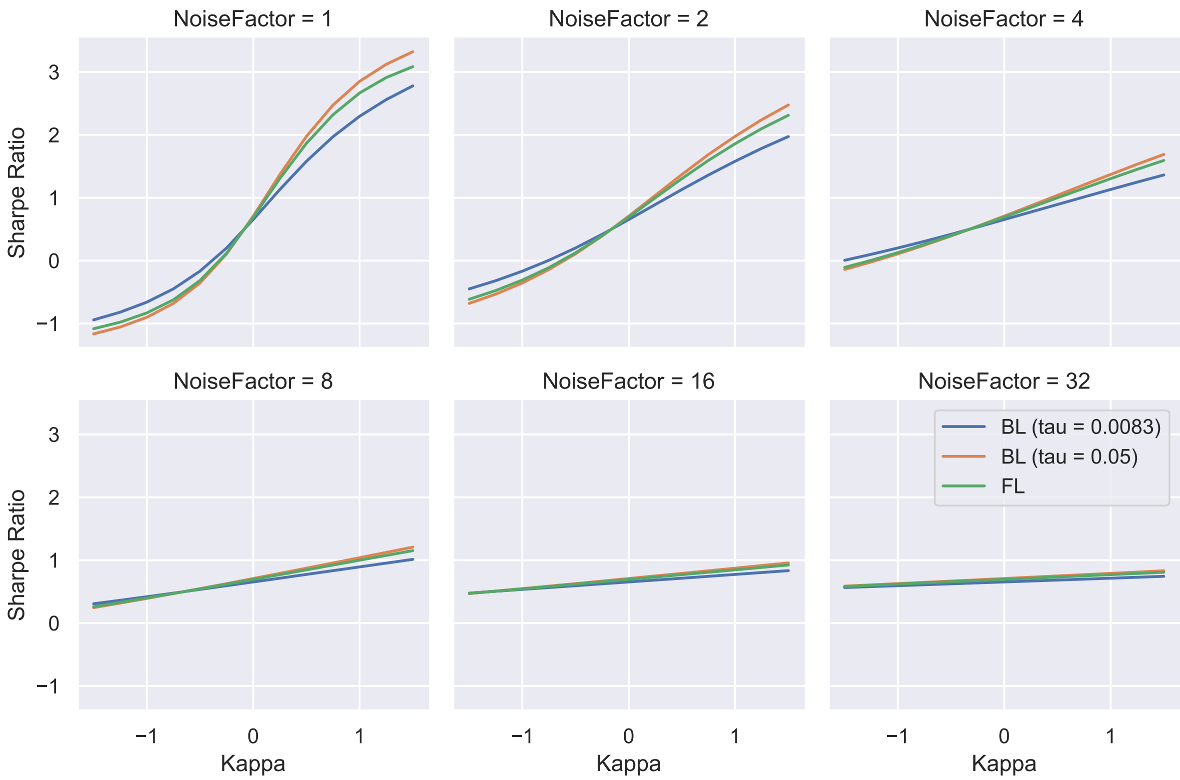

Figure 2: Estimated Sharpe ratios for different models at various points of the noise factor $\breve{\sigma}$ and bias factor $\breve{\kappa}$. Depicted are averages of 300 historical tests with randomized views. The same randomization is used for every model. Relative measures are computed against the MSCI Europe Total Return Index (EUR).

assumed uncertainty about the equilibrium model ($\tau = 0.05$), as it puts more weight on subjective views. The reverse is true in case of bad information. This sensitivity is clearly mitigated in the flexible model, leading to performance that is less responsive to the quality of the views.

The simulation exercises just presented are clearly limited in scope. In our view, the most important contribution of the flexible model is the clear separation of public and private information, that is, between the equilibrium model and subjective views. Recall that in all models studied, for every view expressed by the investor, the uncertainty parameter $\omega_{k,k}$ is specified as well. This parameter conveys exactly how much confidence the investor has in her view. But as emphasized in Figure 3, in the classical Black-Litterman model this information is overlaid with the non-trivial and consequential choice of τ, which again asks how much confidence to put into the views (relative to the equilibrium model). This overlay and interference of two specifications of uncertainty is eliminated in the flexible model. It suffices to specify the view and the associated uncertainty, everything else is estimated objectively from the data.

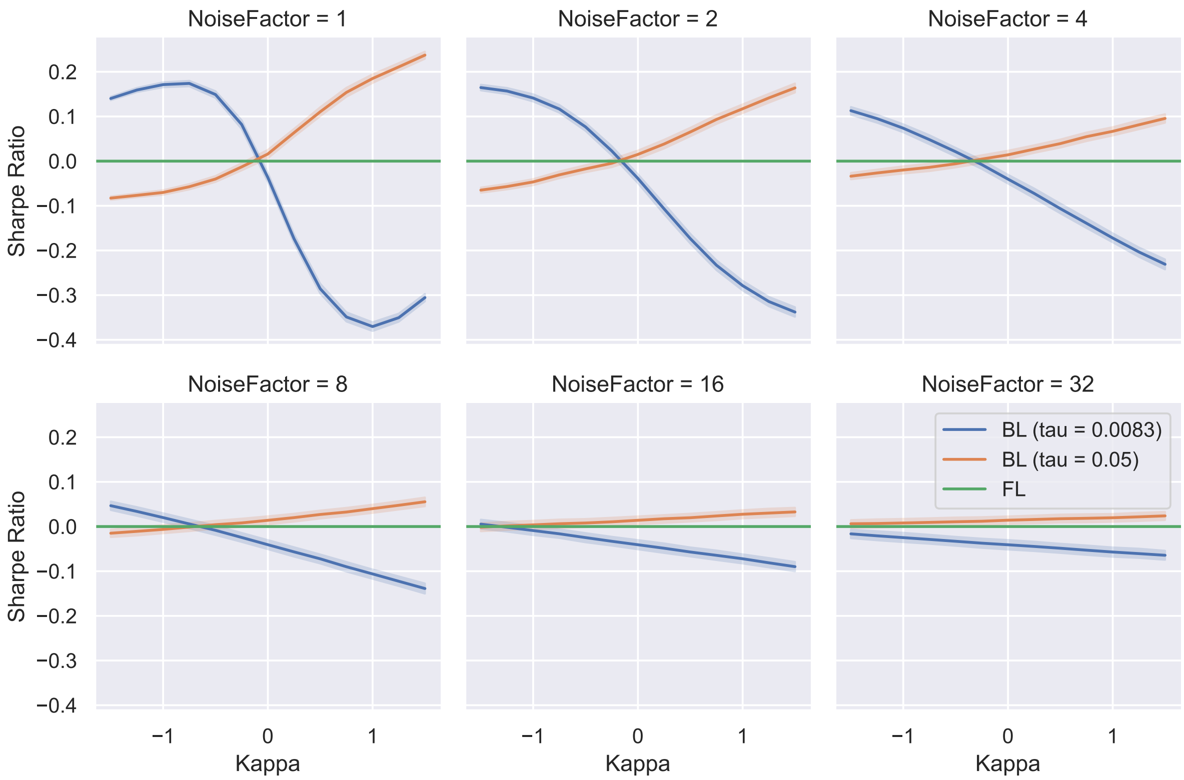

Figure 3: Differences between estimated Sharpe ratios for the flexible model and different models at various points of the noise factor $\breve{\sigma}$ and bias factor $\breve{\kappa}$. Depicted are averages and 95% confidence bands of 300 historical tests with randomized views. The same randomization is used for every model. Relative measures are computed against the MSCI Europe Total Return Index (EUR).

5.5 Robustness Checks

To evaluate the robustness of the results in the preceding sections, we conduct a number of additional historical analysis with different settings. First, we start by comparing the flexible model to a specification proposed by Gofman and Manela (2012), which is in certain aspects similar to our model. We then continue by changing the rebalancing frequency from monthly to quarterly, and test if the same results are achieved when using a 12 year or 8 year lookback window.

Gofman and Manela (2012) In their article, Gofman and Manela (2012) make simplifying assumptions in the Black-Litterman model which lead to the elimination of τ from the model. Instead of using $\boldsymbol{\Psi} = \tau\boldsymbol{\Sigma}$, the authors propose to set $\boldsymbol{\Psi} = \boldsymbol{S}$, i.e. the uncertainty about expected equilibrium returns is equivalent to the idiosyncratic risk which is the residual (or conditional) covariance matrix. This idea is also investigated by Allaj (2013) and complies with the assumption of semi-strong-form market efficiency. It is, in fact, at the heart of the estimation procedure of Allaj (2013) and the reason why his estimates are not comparable to ours[10]. It thus merits a comparison against the flexible model. Table 9 reports performance characteristics of the models, with a direct comparison in the last column. The flexible model is superior in all absolute and relative performance characteristics, indicating that additional information obtained in the stratified second stage of the flexible model conveys important additional information.

Performance Characteristics: Gofman-Manela

	BL Gof.-Man.	FL	FL vs. BL Gof.-Man.
Return p.a.	9.00%	9.34%	
Volatility p.a.	13.52%	13.25%	
Sharpe Ratio	0.67	0.71	
relative Return p.a.	−0.17%	0.15%	0.32%
Tracking Error p.a.	0.46%	0.84%	0.69%
Information Ratio	−0.37	0.18	0.46
Regression Beta	1.01**	0.99**	0.98**
Std. Dev. Beta	0.003	0.006	0.006
Regression Alpha p.a.	−0.24%*	0.27%	0.50%**
T-Stat. of Alpha	−1.71	1.05	2.43
p-Val. of Alpha	0.088	0.296	0.015

Notes: Data used to estimate backtests covers the period 2000-12-31 to 2021-07-31 at a monthly frequency, yielding 248 observations. With a monthly sliding-window estimation, a total of 127 observations of the performance at a monthly frequency are available. Results report characteristics of these time series. The specification labeled as *BL Gof.-Man.* is as outlined in the article Gofman and Manela (2012). Relative measures are computed against the MSCI Europe Total Return Index (EUR), except for the last column, where they are computed against the Gofman-Manela model. The performed regressions are specified as in Equation (38).

* Significant at the 10 percent level.

** Significant at the 5 percent level.

Source: Own Calculation. Data: MSCI Inc.

Table 9: Comparison of performance characteristics of the Gofman-Manela specification and the flexible model.

Quarterly Rebalancing In Table 10 we report performance characteristics of exactly the same models as in Table 5, but using a quarterly instead of a monthly rebalancing frequency. All

[10] Allaj (2013) estimates τ as the proportionality of $\boldsymbol{\Psi}$ to $\boldsymbol{S}$, while in our model we estimate individual τ_i as the proportionality of the diagonal elements of $\boldsymbol{\Psi}$ to $\boldsymbol{\Sigma}$.

essential results are reproduced: The Black-Litterman specifications exhibit negative relative returns while the flexible model achieves positive. Also, the significant alpha of 0.44% is reproduced (vs. 0.45% for monthly rebalancing). We thus conclude that our results are not sensitive to the specification of the rebalancing frequency.

PERFORMANCE CHARACTERISTICS: QUARTERLY REBALANCING

	Market Capitalization	BL ($\tau = \frac{1}{T}$)	BL ($\tau = 0.05$)	FL	FL vs. BL ($\tau = \frac{1}{T}$)
Return p.a.	9.08%	8.91%	8.98%	9.20%	
Volatility p.a.	13.45%	13.51%	13.28%	13.27%	
Sharpe Ratio	0.68	0.66	0.68	0.70	
relative Return p.a.	0.01%	−0.14%	−0.08%	0.13%	0.27%
Tracking Error p.a.	0.04%	0.42%	0.56%	0.84%	0.66%
Information Ratio	0.32	−0.34	−0.14	0.15	0.41
Regression Beta	1.00**	1.00**	0.99**	0.98**	0.98**
Std. Dev. Beta	0.000	0.003	0.004	0.006	0.006
Regression Alpha p.a.	0.02%	−0.18%	0.04%	0.26%	0.44%**
T-Stat. of Alpha	1.14	−1.33	0.25	1.01	2.19
p-Val. of Alpha	0.254	0.184	0.803	0.311	0.029

Notes: Data used to estimate backtests covers the period 2000-12-31 to 2021-06-30 at a monthly frequency, yielding 247 observations. Here, a three month sliding window is used to estimate the models. A total of 126 observations of the performance at a monthly frequency are available. Results report characteristics of these time series. Relative measures are computed against the MSCI Europe Total Return Index (EUR), except for the last column, where they are computed against the Black-Litterman model with $\tau = \frac{1}{T}$. The performed regressions are specified as in Equation (38).

* Significant at the 10 percent level.

** Significant at the 5 percent level.

Source: Own Calculation. Data: MSCI Inc.

Table 10: Comparison of performance characteristics from backtests of different models at a quarterly rebalancing frequency.

Length of Lookback Window: 12 or 8 years Tables 11 and 12 report performance characteristics of the same models as in Table 5, but with varying lookback windows.

First, for the configuration with a 12 year lookback window, a familiar pattern emerges. We again see a negative relative performance of the classical Black-Litterman specifications, and a small but positive relative return of the flexible model. Table 11 also reports a significant alpha when comparing directly the classical Black-Litterman model to the flexible model.

Second, in Table 12, results for an 8 year lookback window. Here, we see slightly different results: While the relative performance of the flexible model still is the best, the two classical Black-Litterman specifications also achieve positive relative returns. Besides the shorter look-back window, this could also be caused by the longer historical backtest (as two more years of data are available to perform it). Also, we still find an alpha of 0.40% for the flexible model against the classical Black-Litterman model, however, it is no longer statistically significant. From these results, we conclude that the results are sensitive to the lookback window in so far as a sufficient amount of data is required to reliably estimate the model parameters, especially for the stratified second stage of the flexible model. When the time window is only 8 years, results are similar in quality, however they lack statistical significance. We would thus suggest to use a lookback window of around 10 years for estimation of the flexible model.

Performance Characteristics: 12 year Lookback

	Market Capitalization	BL ($\tau = \frac{1}{T}$)	BL ($\tau = 0.05$)	FL	FL vs. BL ($\tau = \frac{1}{T}$)
Return p.a.	10.06%	9.84%	9.93%	10.15%	
Volatility p.a.	13.39%	13.55%	13.30%	13.38%	
Sharpe Ratio	0.77	0.74	0.77	0.78	
relative Return p.a.	−0.01%	−0.21%	−0.13%	0.07%	0.28%
Tracking Error p.a.	0.02%	0.46%	0.51%	0.70%	0.51%
Information Ratio	−0.46	−0.45	−0.25	0.11	0.55
Regression Beta	1.00**	1.01**	0.99**	1.00**	0.99**
Std. Dev. Beta	0.000	0.005	0.004	0.005	0.003
Regression Alpha p.a.	−0.01%	−0.32%**	−0.05%	0.10%	0.41%**
T-Stat. of Alpha	−1.20	−1.98	−0.27	0.41	2.43
p-Val. of Alpha	0.230	0.047	0.788	0.685	0.015

Notes: Data used to estimate backtests covers the period 2000-12-31 to 2021-07-31 at a monthly frequency, yielding 248 observations. For each period, a sliding window of data is used to estimate the optimal allocation, each window covering $T = 145$ observations, with a monthly rebalancing frequency. Optimal allocations are thus available for the period 2012-12-31 to 2021-07-31 for a total of 103 observations of the performance at a monthly frequency. Results report characteristics of these time series. Relative measures are computed against the MSCI Europe Total Return Index (EUR), except for the last column, where they are computed against the Black-Litterman model with $\tau = \frac{1}{T}$. The performed regressions are specified as in Equation (38).

* Significant at the 10 percent level.

** Significant at the 5 percent level.

Source: Own Calculation. Data: MSCI Inc.

Table 11: Comparison of performance characteristics from backtests of different models with a 12 year lookback window.

The various robustness tests conducted generally confirm the favourable results reported in Section 5.3 for the performance of the equilibrium part of the proposed flexible model. The model exploits additional information, not taken into account by the proposition of Gofman and Manela (2012), and does this also under varying settings.

Performance Characteristics: 8 year Lookback

	Market Capitalization	BL ($\tau = \frac{1}{T}$)	BL ($\tau = 0.05$)	FL	FL vs. BL ($\tau = \frac{1}{T}$)
Return p.a.	9.82%	9.77%	9.90%	9.99%	
Volatility p.a.	13.27%	13.33%	13.07%	13.08%	
Sharpe Ratio	0.74	0.73	0.76	0.76	
relative Return p.a.	0.06%	0.02%	0.14%	0.22%	0.21%
Tracking Error p.a.	0.09%	0.41%	0.60%	1.04%	0.88%
Information Ratio	0.71	0.04	0.24	0.22	0.24
Regression Beta	1.00**	1.00**	0.98**	0.98**	0.98**
Std. Dev. Beta	0.001	0.003	0.006	0.009	0.008
Regression Alpha p.a.	0.07%	−0.01%	0.30%	0.39%	0.40%
T-Stat. of Alpha	2.66	−0.10	1.54	1.19	1.46
p-Val. of Alpha	0.008	0.920	0.123	0.235	0.145

Notes: Data used to estimate backtests covers the period 2000-12-31 to 2021-07-31 at a monthly frequency, yielding 248 observations. For each period, a sliding window of data is used to estimate the optimal allocation, each window covering $T = 97$ observations, with a monthly rebalancing frequency. Optimal allocations are thus available for the period 2008-12-31 to 2021-07-31 for a total of 151 observations of the performance at a monthly frequency. Results report characteristics of these time series. Relative measures are computed against the MSCI Europe Total Return Index (EUR), except for the last column, where they are computed against the Black-Litterman model with $\tau = \frac{1}{T}$. The performed regressions are specified as in Equation (38).

* Significant at the 10 percent level.

** Significant at the 5 percent level.

Source: Own Calculation. Data: MSCI Inc.

Table 12: Comparison of performance characteristics from backtests of different models with an 8 year lookback window.

6 Conclusion

In this paper, we propose a new model to characterise the equilibrium in the Black-Litterman model. It solves a major limitation of the classical Black-Litterman model, namely the requirement for calibration of the equilibrium uncertainty parameter. As the new model estimates this uncertainty from the data, the entire distribution of the equilibrium is observable and thus becomes public information. In contrast to the classical Black-Litterman model, this allows the new model to comply with the assumption of semi-strong-form market efficiency.

We apply the new model empirically to solve a regional allocation problem for European equities and find that the equilibrium allocation is better diversified and results in better historical performance than in the classical Black-Litterman models. Compared to the most common calibration of the classical model, a statistically significant Jensen's Alpha results, at an annualized 0.45%. Around 60% of this alpha arises due to the estimation of equilibrium uncertainty and 40% stems from increased flexibility. Additionally, when subjective views are introduced through a simulation study, we show that the new model achieves stable results, with a performance much less sensitive to the quality of the views than in the classical model.

Our results are specific to a regional asset allocation exercise. This is clearly a limitation. It would be interesting to see how the model performance under different circumstances. Especially incorporating the procedure of Phillips (2003) and applying the model also to a stock selection exercise could yield insightful results. While we extended the simulation study of Gofman and Manela (2012) slightly, the evaluation in the presence of views is also still a topic worth pursuing.

Still, our model offers practitioners a new tool to improve the asset allocation process by focusing on the specification of their views, without the need for parametrization of the equilibrium model.

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